PE3101P: Decision Theory and Social Choice

Take-Home Assignment 1: Model Solutions

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Question 1

Recall that, for any two acts A and B over the same set of states of nature:

- A **strongly statewise dominates** B if A results in a better outcome than B in *every* state of nature.
- A **weakly statewise dominates** B if A results in an outcome at least as good as B's in *every* state of nature, and a better outcome in *at least one* state of nature.

(a) Show that if A strongly statewise dominates B, and B strongly statewise dominates C, then A strongly statewise dominates C.

(b) Show that the same holds for weak statewise dominance: if A weakly statewise dominates B, and B weakly statewise dominates C, then A weakly statewise dominates C.

(c) Now suppose only one of the two links is strong. Show that if A strongly statewise dominates B and B weakly statewise dominates C, then A **strongly** statewise dominates C; and that the same conclusion follows if instead A weakly statewise dominates B and B strongly statewise dominates C.

Solution

Write $A(s)$ for the outcome of act A in state s. Write $\succeq$ for "at least as good as" and $\succ$ for "better than" between outcomes.

(a) Suppose A strongly statewise dominates B, and B strongly statewise dominates C. Let s be any state. Then

$$A(s) \succ B(s) \quad \text{and} \quad B(s) \succ C(s).$$

By transitivity of betterness, $A(s) \succ C(s)$. Since s was arbitrary, this holds in every state. Thus A strongly statewise dominates C.

(b) Suppose A weakly statewise dominates B, and B weakly statewise dominates C. We establish both requirements for weak dominance.

First, in every state s, $A(s) \succeq B(s)$ and $B(s) \succeq C(s)$. Transitivity therefore gives $A(s) \succeq C(s)$ in every state.

Second, weak dominance of B by A gives a state s^* such that $A(s^*) \succ B(s^*)$. Since B weakly statewise dominates C, we also have $B(s^*) \succeq C(s^*)$. Hence $A(s^*) \succ C(s^*)$ by transitivity.

Thus A is better than C in at least one state, as well as at least as good in every state. Hence A weakly statewise dominates C.

(c) There are two cases. If A strongly dominates B and B weakly dominates C, then in every state s,

$$A(s) \succ B(s) \succeq C(s), \qquad \text{so } A(s) \succ C(s).$$

If A weakly dominates B and B strongly dominates C, then in every state s,

$$A(s) \succeq B(s) \succ C(s), \qquad \text{so } A(s) \succ C(s).$$

In either case, A strongly statewise dominates C.

Question 2

In each of the following, exactly one of the four options denotes the same event as the expression at the top, whatever the events A, B and C happen to be. Say which.

Throughout, X^c is the complement of X, and $X \setminus Y$ is the set of outcomes that are in X but not in Y.

(a) $(A \cup B)^c$

- (i) $A^c \cup B^c$
- (ii) $A^c \cap B^c$
- (iii) $A^c \cup B$
- (iv) $(A \cap B)^c$

(b) $A \cap (B \cup C)$

- (i) $(A \cup B) \cap (A \cup C)$
- (ii) $(A \cap B) \cup C$
- (iii) $(A \cap B) \cup (A \cap C)$
- (iv) $A \cap B \cap C$

(c) $A \setminus (B \cup C)$

- (i) $(A \setminus B) \cup (A \setminus C)$
- (ii) $(A \cup B) \setminus C$
- (iii) $A \setminus (B \cap C)$
- (iv) $(A \setminus B) \cap (A \setminus C)$

(d) $(A \cap B) \cup (A \cap B^c)$

- (i) B
- (ii) A
- (iii) $A \cap B$
- (iv) $A \cup B$

(e) $(A \cup B) \cap (A \cup B^c)$

(i) $A \cup B$
(ii) $\varnothing$
(iii) B
(iv) A

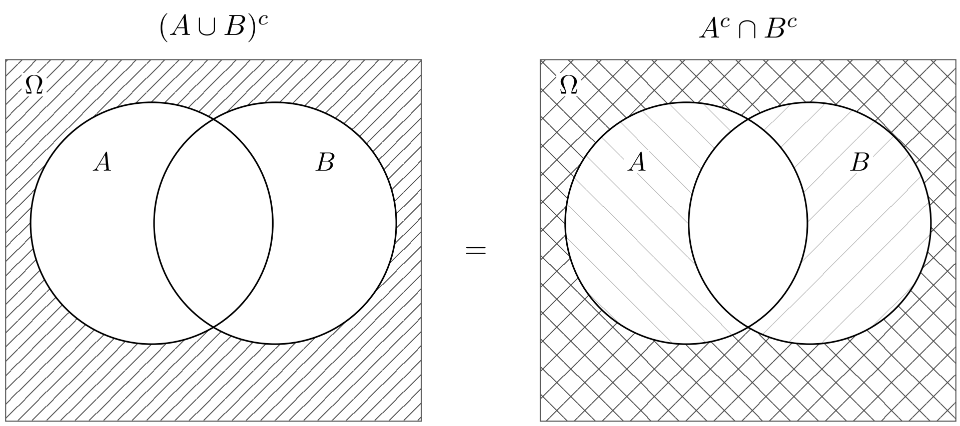
Solution

(a) Option (ii). By De Morgan's law,

$$(A \cup B)^c = A^c \cap B^c.$$

An outcome is outside $A \cup B$ exactly when it is outside both A and B.

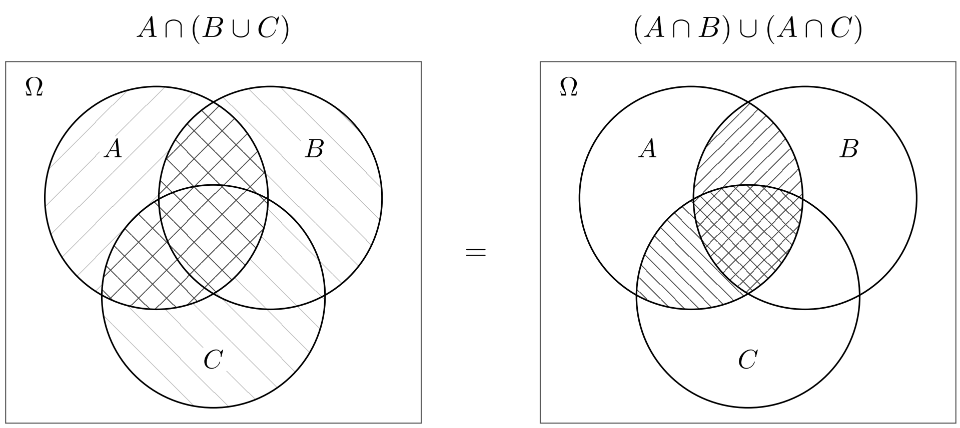
In the diagrams, opposite hatch directions distinguish the two parts of a union. For an intersection, faint hatching shows the two sets; dense hatching marks their common region.



(b) Option (iii). By distributivity,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

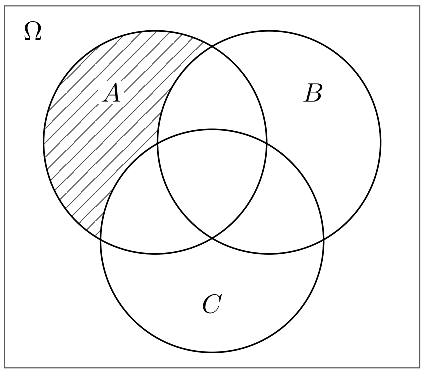
An outcome lies in A and in at least one of B, C exactly when it lies in $A \cap B$ or in $A \cap C$.



(c) **Option (iv).** Using set difference, De Morgan's law, and the associativity and idempotence of intersection,

$$\begin{aligned} A \setminus (B \cup C) &= A \cap (B \cup C)^c \\ &= A \cap B^c \cap C^c \\ &= (A \cap B^c) \cap (A \cap C^c) \\ &= (A \setminus B) \cap (A \setminus C). \end{aligned}$$

$A \setminus (B \cup C)$



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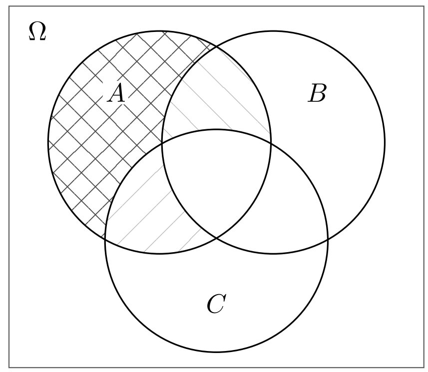
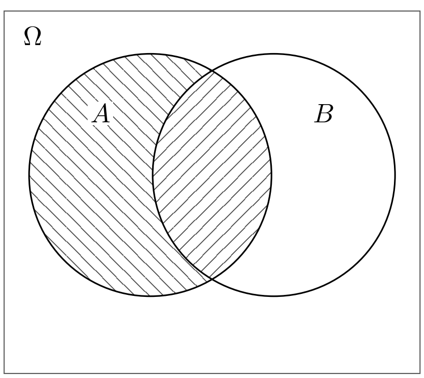
$(A \setminus B) \cap (A \setminus C)$



(d) **Option (ii).** By distributivity and complementation,

$$(A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c) = A \cap \Omega = A.$$

$(A \cap B) \cup (A \cap B^c)$



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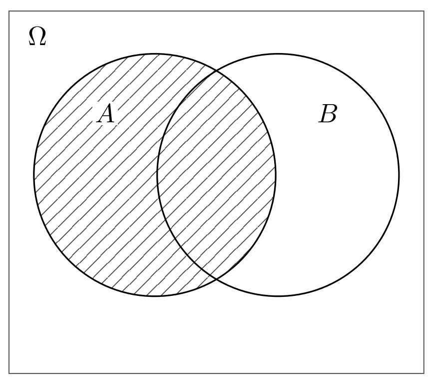
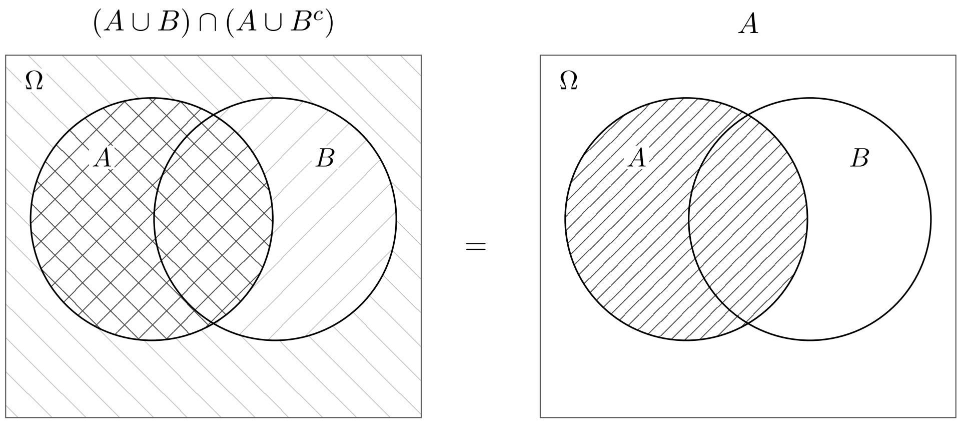


(e) **Option (iv).** By distributivity and complementation,

$$(A \cup B) \cap (A \cup B^c) = A \cup (B \cap B^c) = A \cup \varnothing = A.$$



Only the option choices were required for Question 2. These explanations and diagrams are included for your benefit.

Question 3

Recall the probability axioms. For any events A and B:

- **Non-negativity:** $\mathrm{Cr}(A) \geq 0$.
- **Normalisation:** $\mathrm{Cr}(\Omega) = 1$, where Ω is the sample space.
- **Additivity:** if A and B are disjoint, then $\mathrm{Cr}(A \cup B) = \mathrm{Cr}(A) + \mathrm{Cr}(B)$.

In each of the following, an agent's credences are given. Say whether they can all be held together without violating the axioms. If they cannot, say which axiom or axioms are violated, and explain why. If one of two or more axioms must be violated, but it is not settled which one, indicate this clearly.

(a) A commuter is uncertain whether the 8:15 bus will arrive on time.

$$\mathrm{Cr}(\text{the bus is on time}) = 0.7, \qquad \mathrm{Cr}(\text{the bus is not on time}) = 0.4.$$

(b) A ball is to be drawn from a bag containing balls of several colours, some of which are plastic.

$$\mathrm{Cr}(\text{the ball is red}) = 0.2, \qquad \mathrm{Cr}(\text{the ball is red and plastic}) = 0.35.$$

(c) A fair spinner is divided into eight equal sectors, numbered 1 to 8.

$$\mathrm{Cr}(\text{even}) = 1/2, \quad \mathrm{Cr}(\text{lands on } 3) = 1/8, \quad \mathrm{Cr}(\text{even, or lands on } 3) = 5/8.$$

(d) A football match is to be played. It can end in a home win, an away win, or a draw.

$$\mathrm{Cr}(\text{a home win}) = 0.5, \quad \mathrm{Cr}(\text{a draw}) = 0.2, \quad \mathrm{Cr}(\text{a home win, or a draw}) = 0.8.$$

(e) A student is taking two modules this semester.

$$\mathrm{Cr}(\text{passes the first}) = 0.8, \quad \mathrm{Cr}(\text{passes the second}) = 0.7, \quad \mathrm{Cr}(\text{passes at least one}) = 0.9.$$

Solution

(a) Inconsistent: at least one of additivity and normalisation must fail. Let T be the event that the bus is on time. Since T and T^c are disjoint, additivity requires

$$\mathrm{Cr}(\Omega) = \mathrm{Cr}(T \cup T^c) = \mathrm{Cr}(T) + \mathrm{Cr}(T^c) = 0.7 + 0.4 = 1.1.$$

Normalisation instead requires $\mathrm{Cr}(\Omega) = 1$. The stated credences cannot satisfy both axioms; they do not settle which one fails.

(b) Inconsistent: at least one of additivity and non-negativity must fail. Let R be the event that the ball is red and P that it is plastic. The disjoint events $R \cap P$ and $R \setminus P$ have union R. Additivity would therefore require

$$\mathrm{Cr}(R) = \mathrm{Cr}(R \cap P) + \mathrm{Cr}(R \setminus P).$$

Rearranging gives

$$\mathrm{Cr}(R \setminus P) = \mathrm{Cr}(R) - \mathrm{Cr}(R \cap P) = 0.2 - 0.35 = -0.15.$$

This contradicts non-negativity. The stated credences cannot satisfy both axioms; they do not settle which one fails.

(c) Consistent. Assign credence 1/8 to each of the eight sectors. This satisfies the axioms and gives

$$\mathrm{Cr}(\text{even}) = \frac{4}{8} = \frac{1}{2}, \quad \mathrm{Cr}(3) = \frac{1}{8}, \quad \mathrm{Cr}(\text{even or } 3) = \frac{5}{8},$$

since the even sectors and sector 3 are disjoint.

You were not required to demonstrate consistency here or in part (e). In general, one can do so by giving a sample space and a credence function satisfying both the stated conditions and the axioms. Here, take the eight sectors as outcomes and define an event's credence as the sum of the credences of its outcomes. The eight non-negative weights sum to 1, giving non-negativity and normalisation; summing over disjoint events also gives additivity.

(d) Inconsistent: additivity fails. A home win W and a draw D are disjoint. Additivity requires

$$\mathrm{Cr}(W \cup D) = \mathrm{Cr}(W) + \mathrm{Cr}(D) = 0.5 + 0.2 = 0.7,$$

whereas the given credence is 0.8.

(e) Consistent. Let A, B be passing the first and second modules. Take a sample space with just four outcomes: passing both modules, passing only the first, passing only the second, and passing neither. Assign them the following weights:

Event	$A \cap B$	$A \cap B^c$	$A^c \cap B$	$A^c \cap B^c$
Credence	0.6	0.2	0.1	0.1

Define each event's credence as the sum of the weights of its outcomes. The weights are non-negative and sum to 1, and this definition satisfies additivity. Adding the appropriate entries gives $\mathrm{Cr}(A) = 0.8$, $\mathrm{Cr}(B) = 0.7$ and $\mathrm{Cr}(A \cup B) = 0.9$, as required.

Question 4

Let Cr be a credence function satisfying the probability axioms. Suppose the sample space Ω can be partitioned into $B_1, B_2, \ldots, B_n$ — that is, the B_i are mutually exclusive and jointly exhaustive. Let A be any event.

(a) Show that

$$\mathrm{Cr}(A) = \mathrm{Cr}(A \cap B_1) + \mathrm{Cr}(A \cap B_2) + \cdots + \mathrm{Cr}(A \cap B_n).$$

You may assume the following, which you are not required to prove: if B and C are disjoint, then $A \cap B$ and $A \cap C$ are disjoint. You may also assume that $A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$.

(b) Hence prove the **law of total probability**: where $\mathrm{Cr}(B_i) > 0$ for every i,

$$\mathrm{Cr}(A) = \mathrm{Cr}(B_1)\,\mathrm{Cr}(A \mid B_1) + \mathrm{Cr}(B_2)\,\mathrm{Cr}(A \mid B_2) + \cdots + \mathrm{Cr}(B_n)\,\mathrm{Cr}(A \mid B_n).$$

Solution

(a) Since the B_i are mutually exclusive, $B_i \cap B_j = \varnothing$ whenever $i \neq j$. By the first granted fact, $A \cap B_i$ and $A \cap B_j$ are therefore disjoint whenever $i \neq j$.

By the second granted fact,

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n).$$

Applying additivity to these disjoint events gives

$$\begin{aligned}\mathrm{Cr}(A) &= \mathrm{Cr}\big((A \cap B_1) \cup \cdots \cup (A \cap B_n)\big) \\ &= \mathrm{Cr}(A \cap B_1) + \cdots + \mathrm{Cr}(A \cap B_n).\end{aligned}$$

Or, if you prefer,

$$\mathrm{Cr}(A) = \sum_{i=1}^{n} \mathrm{Cr}(A \cap B_i).$$

The repeated use of additivity works because the union of the first k events is disjoint from $A \cap B_{k+1}$. You were not required to state or prove this additional observation.

(b) For each i, $\mathrm{Cr}(B_i) > 0$, so the definition of conditional probability gives

$$\mathrm{Cr}(A \mid B_i) = \frac{\mathrm{Cr}(A \cap B_i)}{\mathrm{Cr}(B_i)}.$$

Multiplying by $\mathrm{Cr}(B_i)$,

$$\mathrm{Cr}(A \cap B_i) = \mathrm{Cr}(B_i)\,\mathrm{Cr}(A \mid B_i).$$

Substitute this equality for each term in part (a):

$$\mathrm{Cr}(A) = \mathrm{Cr}(B_1)\,\mathrm{Cr}(A \mid B_1) + \cdots + \mathrm{Cr}(B_n)\,\mathrm{Cr}(A \mid B_n).$$

Or, if you prefer,

$$\boxed{\mathrm{Cr}(A) = \sum_{i=1}^{n} \mathrm{Cr}(B_i)\,\mathrm{Cr}(A \mid B_i).}$$

This is the law of total probability.

Question 5

Recall the definition of conditional probability: for any events X and Y,

$$\mathrm{Cr}(X \mid Y) = \frac{\mathrm{Cr}(X \cap Y)}{\mathrm{Cr}(Y)}.$$

Assume throughout this question that every event mentioned has positive credence.

(a) Using only this definition, prove **Bayes' theorem**:

$$\mathrm{Cr}(A \mid B) = \frac{\mathrm{Cr}(B \mid A) \cdot \mathrm{Cr}(A)}{\mathrm{Cr}(B)}.$$

(b) Using your answer to part (a), show that

$$\underbrace{\frac{\mathrm{Cr}(H_1 \mid E)}{\mathrm{Cr}(H_2 \mid E)}}_{\text{posterior odds}} = \underbrace{\frac{\mathrm{Cr}(E \mid H_1)}{\mathrm{Cr}(E \mid H_2)}}_{\text{Bayes factor}} \times \underbrace{\frac{\mathrm{Cr}(H_1)}{\mathrm{Cr}(H_2)}}_{\text{prior odds}}.$$

Solution

(a) By the definition of conditional probability,

$$\mathrm{Cr}(A \mid B) = \frac{\mathrm{Cr}(A \cap B)}{\mathrm{Cr}(B)}, \qquad \mathrm{Cr}(B \mid A) = \frac{\mathrm{Cr}(B \cap A)}{\mathrm{Cr}(A)}.$$

Since $B \cap A = A \cap B$, the second equality gives

$$\mathrm{Cr}(A \cap B) = \mathrm{Cr}(B \mid A)\,\mathrm{Cr}(A).$$

Substituting into the first equality yields Bayes' theorem:

$$\boxed{\mathrm{Cr}(A \mid B) = \frac{\mathrm{Cr}(B \mid A)\,\mathrm{Cr}(A)}{\mathrm{Cr}(B)}.}$$

(b) Apply part (a) to H_1 and H_2, each conditional on E:

$$\mathrm{Cr}(H_1 \mid E) = \frac{\mathrm{Cr}(E \mid H_1)\,\mathrm{Cr}(H_1)}{\mathrm{Cr}(E)},$$

$$\mathrm{Cr}(H_2 \mid E) = \frac{\mathrm{Cr}(E \mid H_2)\,\mathrm{Cr}(H_2)}{\mathrm{Cr}(E)}.$$

Dividing the first equality by the second gives

$$\begin{aligned}
\frac{\mathrm{Cr}(H_1 \mid E)}{\mathrm{Cr}(H_2 \mid E)} &= \frac{\mathrm{Cr}(E \mid H_1)\,\mathrm{Cr}(H_1)/\,\mathrm{Cr}(E)}{\mathrm{Cr}(E \mid H_2)\,\mathrm{Cr}(H_2)/\,\mathrm{Cr}(E)} \\
&= \frac{\mathrm{Cr}(E \mid H_1)\,\mathrm{Cr}(H_1)}{\mathrm{Cr}(E \mid H_2)\,\mathrm{Cr}(H_2)} && \text{(cancel the common divisor } \mathrm{Cr}(E)) \\
&= \frac{\mathrm{Cr}(E \mid H_1)}{\mathrm{Cr}(E \mid H_2)} \times \frac{\mathrm{Cr}(H_1)}{\mathrm{Cr}(H_2)}.
\end{aligned}$$

Hence

$$\boxed{\text{posterior odds} = \text{Bayes factor} \times \text{prior odds.}}$$

Question 6

A bag contains three six-sided dice, which look and feel identical. One of them is loaded, and lands six with probability $1/2$; the other two are fair. You take one die from the bag at random, roll it once, and it lands six.

Let H_1 be the hypothesis that you took the loaded die, H_2 the hypothesis that you took a fair one, and let E be the evidence that the roll landed six.

(a) Write down the prior odds of H_1 over H_2, and the Bayes factor.

(b) Using the odds form of Bayes' theorem, as stated in question 5(b), calculate the posterior odds, and hence your credence in H_1 and in H_2 after seeing the roll.

(c) Suppose instead that the roll had *not* landed six, and everything else had gone as before. What would your credence in H_1 have been? Which of the prior odds, the Bayes factor and the posterior odds changed, and which did not?

Solution

(a) One of the three dice is loaded, and two are fair. Therefore

$$\mathrm{Cr}(H_1) = \frac{1}{3}, \qquad \mathrm{Cr}(H_2) = \frac{2}{3}, \qquad \frac{\mathrm{Cr}(H_1)}{\mathrm{Cr}(H_2)} = \frac{1}{2}.$$

The loaded die has probability 1/2 of rolling six; a fair six-sided die has probability 1/6. For the evidence E that the roll is six, the Bayes factor is therefore

$$\frac{\mathrm{Cr}(E \mid H_1)}{\mathrm{Cr}(E \mid H_2)} = \frac{1/2}{1/6} = 3.$$

(b) The posterior odds of H_1 over H_2 are

$$\frac{\mathrm{Cr}(H_1 \mid E)}{\mathrm{Cr}(H_2 \mid E)} = 3 \times \frac{1}{2} = \frac{3}{2}.$$

The hypotheses are mutually exclusive and jointly exhaustive, so their posterior credences sum to 1. Writing $p = \mathrm{Cr}(H_1 \mid E)$, we have

$$\frac{p}{1-p} = \frac{3}{2} \implies 2p = 3(1-p) \implies p = \frac{3}{5}.$$

Thus $\boxed{\mathrm{Cr}(H_1 \mid E) = 3/5}$ and $\boxed{\mathrm{Cr}(H_2 \mid E) = 2/5}$.

(c) Suppose instead that this single roll is not six. The evidence is E^c, with conditional probabilities

$$\mathrm{Cr}(E^c \mid H_1) = 1 - \frac{1}{2} = \frac{1}{2}, \qquad \mathrm{Cr}(E^c \mid H_2) = 1 - \frac{1}{6} = \frac{5}{6}.$$

The Bayes factor is now $(1/2)/(5/6) = 3/5$. The prior odds remain $1/2$, so the posterior odds are $(3/5)(1/2) = 3/10$. Since H_1, H_2 form a partition, they are complements: additivity and normalisation make their posterior credences sum to 1. Writing $p = \mathrm{Cr}(H_1 \mid E^c)$, we have

$$\frac{p}{1-p} = \frac{3}{10} \quad \Longrightarrow \quad 10p = 3(1-p) \quad \Longrightarrow \quad 13p = 3 \quad \Longrightarrow \quad p = \frac{3}{13}.$$

Thus $\boxed{\mathrm{Cr}(H_1 \mid E^c) = 3/13}$ and $\boxed{\mathrm{Cr}(H_2 \mid E^c) = 10/13}$.

Quantity	Six	Not six
Prior odds	1/2	1/2
Bayes factor	3	3/5
Posterior odds	3/2	3/10

Only the prior odds are unchanged. A non-six favours a fair die, reducing the credence in H_1 from its prior value 1/3 to 3/13.

Question 7

In a decision problem with finitely many acts and finitely many possible outcomes, a **dependency hypothesis** lists, for each act available, which outcome would result were that act performed.

It will help to have a compact notation. Where there are two acts, write $\langle o, o' \rangle$ for the dependency hypothesis on which the first act would produce outcome o and the second would produce outcome o'; and similarly, with a third entry, where there are three acts. Each outcome below is given a letter to use in place of its name.

For example, take the acts *revise* and *go out*, with outcomes *pass* (p) and *fail* (f). One of the dependency hypotheses is $\langle p, f \rangle$: revising would result in passing, and going out would result in failing.

(a) In each of the following, list **every** possible dependency dependency hypothesis.

- (i) Acts: *take the umbrella*, *leave the umbrella*. Outcomes: *dry* (d), *soaked* (s).
- (ii) Acts: *water the plant*, *leave the plant*. Outcomes: *flowers* (f), *survives* (s), *dies* (d).
- (iii) Acts: *brush the cat*, *stroke the cat*, *leave the cat alone*. Outcomes: *purrs* (p), *quiet* (q).

(b) In general, how many dependency hypotheses are there when a decision problem has m acts and n possible outcomes? Explain your answer.

Solution

(a) Each entry records the outcome of the corresponding act, with acts ordered as on the question sheet.

(i) Umbrella: four dependency hypotheses. The first entry is for taking the umbrella; the second is for leaving it.

$$\langle d, d \rangle, \quad \langle d, s \rangle, \quad \langle s, d \rangle, \quad \langle s, s \rangle.$$

(ii) Plant: nine dependency hypotheses. The first entry is for watering the plant; the second is for leaving it.

$$\begin{array}{ccc} \langle f, f \rangle & \langle f, s \rangle & \langle f, d \rangle \\ \langle s, f \rangle & \langle s, s \rangle & \langle s, d \rangle \\ \langle d, f \rangle & \langle d, s \rangle & \langle d, d \rangle \end{array}$$

(iii) Cat: eight dependency hypotheses. The entries are for brushing the cat, stroking it, and leaving it alone, respectively.

$$\begin{array}{cccc} \langle p, p, p \rangle & \langle p, p, q \rangle & \langle p, q, p \rangle & \langle p, q, q \rangle \\ \langle q, p, p \rangle & \langle q, p, q \rangle & \langle q, q, p \rangle & \langle q, q, q \rangle \end{array}$$

To list these systematically, vary the last entry through all its possible values while holding the earlier entries fixed. Then change the preceding entry and repeat, continuing in this way. This avoids missing a dependency hypothesis.

(b) There are $\boxed{n^m}$ dependency hypotheses.

A dependency hypothesis specifies one outcome for each of the m acts. There are n choices for the first entry. For each of these, there are n choices for the second entry, and so on through all m entries. The multiplication rule for counting therefore gives the number of dependency hypotheses:

$$\underbrace{n \times n \times \cdots \times n}_{m \text{ factors}} = n^m.$$