PE3101P: Decision Theory and Social Choice

Take-Home Assignment 1

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Due Tuesday 8 September 2026, 23:59

Instructions

Answer **all** questions. **The paper is marked out of 100.** The marks available are shown against each question, in the right-hand margin.

You may use results proved in the lecture slides, in the lecture notes, or in the primer *Sets and Mathematical Proofs*, and you may use the probability axioms freely, unless a question says otherwise. Where a question asks you to prove something, set the proof out in steps and say what justifies each one.

This assignment is worth **20%** of your final grade for the module.

Question 1 **[15 marks]**

Recall that, for any two acts A and B over the same set of states of nature:

– A **strongly statewise dominates** B if A results in a better outcome than B in *every* state of nature.

– A **weakly statewise dominates** B if A results in an outcome at least as good as B's in *every* state of nature, and a better outcome in *at least one* state of nature.

(a) Show that if A strongly statewise dominates B, and B strongly statewise dominates C, then A strongly statewise dominates C.

(b) Show that the same holds for weak statewise dominance: if A weakly statewise dominates B, and B weakly statewise dominates C, then A weakly statewise dominates C.

(c) Now suppose only one of the two links is strong. Show that if A strongly statewise dominates B and B weakly statewise dominates C, then A **strongly** statewise dominates C; and that the same conclusion follows if instead A weakly statewise dominates B and B strongly statewise dominates C.

Question 2 **[10 marks]**

In each of the following, exactly one of the four options denotes the same event as the expression at the top, whatever the events A, B and C happen to be. Say which.

Throughout, X^c is the complement of X, and $X \setminus Y$ is the set of outcomes that are in X but not in Y.

(a) $(A \cup B)^c$

- (i) $A^c \cup B^c$
- (ii) $A^c \cap B^c$
- (iii) $A^c \cup B$
- (iv) $(A \cap B)^c$

(b) $A \cap (B \cup C)$

- (i) $(A \cup B) \cap (A \cup C)$
- (ii) $(A \cap B) \cup C$
- (iii) $(A \cap B) \cup (A \cap C)$
- (iv) $A \cap B \cap C$

(c) $A \setminus (B \cup C)$

- (i) $(A \setminus B) \cup (A \setminus C)$
- (ii) $(A \cup B) \setminus C$
- (iii) $A \setminus (B \cap C)$
- (iv) $(A \setminus B) \cap (A \setminus C)$

(d) $(A \cap B) \cup (A \cap B^c)$

- (i) B
- (ii) A
- (iii) $A \cap B$
- (iv) $A \cup B$

(e) $(A \cup B) \cap (A \cup B^c)$

- (i) $A \cup B$
- (ii) $\varnothing$
- (iii) B
- (iv) A

Question 3 [**15 marks**]

Recall the probability axioms. For any events A and B:

- **Non-negativity:** $\mathrm{Cr}(A) \geq 0$.
- **Normalisation:** $\mathrm{Cr}(\Omega) = 1$, where Ω is the sample space.
- **Additivity:** if A and B are disjoint, then $\mathrm{Cr}(A \cup B) = \mathrm{Cr}(A) + \mathrm{Cr}(B)$.

In each of the following, an agent's credences are given. Say whether they can all be held together without violating the axioms. If they cannot, say which axiom or axioms are violated, and explain why. If one of two or more axioms must be violated, but it is not settled which one, indicate this clearly.

(a) A commuter is uncertain whether the 8:15 bus will arrive on time.

$$\mathrm{Cr}(\text{the bus is on time}) = 0.7, \qquad \mathrm{Cr}(\text{the bus is not on time}) = 0.4.$$

(b) A ball is to be drawn from a bag containing balls of several colours, some of which are plastic.

$$\mathrm{Cr}(\text{the ball is red}) = 0.2, \qquad \mathrm{Cr}(\text{the ball is red and plastic}) = 0.35.$$

(c) A fair spinner is divided into eight equal sectors, numbered 1 to 8.

$$\mathrm{Cr}(\text{even}) = 1/2, \quad \mathrm{Cr}(\text{lands on 3}) = 1/8, \quad \mathrm{Cr}(\text{even, or lands on 3}) = 5/8.$$

(d) A football match is to be played. It can end in a home win, an away win, or a draw.

$$\mathrm{Cr}(\text{a home win}) = 0.5, \quad \mathrm{Cr}(\text{a draw}) = 0.2, \quad \mathrm{Cr}(\text{a home win, or a draw}) = 0.8.$$

(e) A student is taking two modules this semester.

$$\mathrm{Cr}(\text{passes the first}) = 0.8, \quad \mathrm{Cr}(\text{passes the second}) = 0.7, \quad \mathrm{Cr}(\text{passes at least one}) = 0.9.$$

Question 4 **[15 marks]**

Let Cr be a credence function satisfying the probability axioms. Suppose the sample space Ω can be partitioned into $B_1, B_2, \ldots, B_n$ — that is, the B_i are mutually exclusive and jointly exhaustive. Let A be any event.

(a) Show that

$$\mathrm{Cr}(A) = \mathrm{Cr}(A \cap B_1) + \mathrm{Cr}(A \cap B_2) + \cdots + \mathrm{Cr}(A \cap B_n).$$

You may assume the following, which you are not required to prove: if B and C are disjoint, then $A \cap B$ and $A \cap C$ are disjoint. You may also assume that $A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$.

(b) Hence prove the **law of total probability**: where $\mathrm{Cr}(B_i) > 0$ for every i,

$$\mathrm{Cr}(A) = \mathrm{Cr}(B_1)\,\mathrm{Cr}(A \mid B_1) + \mathrm{Cr}(B_2)\,\mathrm{Cr}(A \mid B_2) + \cdots + \mathrm{Cr}(B_n)\,\mathrm{Cr}(A \mid B_n).$$

Question 5 **[15 marks]**

Recall the definition of conditional probability: for any events X and Y,

$$\mathrm{Cr}(X \mid Y) = \frac{\mathrm{Cr}(X \cap Y)}{\mathrm{Cr}(Y)}.$$

Assume throughout this question that every event mentioned has positive credence.

(a) Using only this definition, prove **Bayes' theorem**:

$$\mathrm{Cr}(A \mid B) = \frac{\mathrm{Cr}(B \mid A) \cdot \mathrm{Cr}(A)}{\mathrm{Cr}(B)}.$$

(b) Using your answer to part (a), show that

$$\underbrace{\frac{\mathrm{Cr}(H_1 \mid E)}{\mathrm{Cr}(H_2 \mid E)}}_{\text{posterior odds}} = \underbrace{\frac{\mathrm{Cr}(E \mid H_1)}{\mathrm{Cr}(E \mid H_2)}}_{\text{Bayes factor}} \times \underbrace{\frac{\mathrm{Cr}(H_1)}{\mathrm{Cr}(H_2)}}_{\text{prior odds}}.$$

Question 6 **[15 marks]**

A bag contains three six-sided dice, which look and feel identical. One of them is loaded, and lands six with probability 1/2; the other two are fair. You take one die from the bag at random, roll it once, and it lands six.

Let H_1 be the hypothesis that you took the loaded die, H_2 the hypothesis that you took a fair one, and let E be the evidence that the roll landed six.

(a) Write down the prior odds of H_1 over H_2, and the Bayes factor.

(b) Using the odds form of Bayes' theorem, as stated in question 5(b), calculate the posterior odds, and hence your credence in H_1 and in H_2 after seeing the roll.

(c) Suppose instead that the roll had *not* landed six, and everything else had gone as before. What would your credence in H_1 have been? Which of the prior odds, the Bayes factor and the posterior odds changed, and which did not?

Question 7 **[15 marks]**

In a decision problem with finitely many acts and finitely many possible outcomes, a **dependency hypothesis** lists, for each act available, which outcome would result were that act performed.

It will help to have a compact notation. Where there are two acts, write $\langle o, o' \rangle$ for the dependency hypothesis on which the first act would produce outcome o and the second would produce outcome o'; and similarly, with a third entry, where there are three acts. Each outcome below is given a letter to use in place of its name.

For example, take the acts *revise* and *go out*, with outcomes *pass* (p) and *fail* (f). One of the dependency hypotheses is $\langle p, f \rangle$: revising would result in passing, and going out would result in failing.

(a) In each of the following, list **every** possible dependency hypothesis.

(i) Acts: *take the umbrella*, *leave the umbrella*. Outcomes: *dry* (d), *soaked* (s).
(ii) Acts: *water the plant*, *leave the plant*. Outcomes: *flowers* (f), *survives* (s), *dies* (d).
(iii) Acts: *brush the cat*, *stroke the cat*, *leave the cat alone*. Outcomes: *purrs* (p), *quiet* (q).

(b) In general, how many dependency hypotheses are there when a decision problem has m acts and n possible outcomes? Explain your answer.