PE3101P: Decision Theory and Social Choice

In-class Test 1 — Solutions

Time allowed: 90 minutes. Total: 100 marks.

- Section A: answer all 10 multiple-choice questions. Each is worth 4 marks (40 marks in total).
- Section B: answer at least three of the four longer questions. Your best three marks will be counted. Each is worth 20 marks (60 marks in total).

Filled-in answers are shown in **bold**.

Section A

Answer all 10 questions. Each is worth 4 marks. Choose one answer, A–D, for each question, and write its letter in the grid below.

Question	1	2	3	4	5	6	7	8	9	10
Answer	**D**	**A**	**D**	**A**	**D**	**B**	**B**	**C**	**A**	**D**

1. An agent's credences satisfy the probability axioms. She assigns $\mathrm{Cr}(H) = 0.5$, $\mathrm{Cr}(E) = 0.4$, $\mathrm{Cr}(E \mid H) = 0.2$. What is $\mathrm{Cr}(H \mid E)$?

A. 0.1 **B.** 0.75

C. 0.2 **D.** 0.25

Solution. **D: 0.25.** By Bayes' theorem,

$$\mathrm{Cr}(H \mid E) = \frac{\mathrm{Cr}(E \mid H)\ \mathrm{Cr}(H)}{\mathrm{Cr}(E)} = \frac{0.2 \times 0.5}{0.4} = \frac{0.10}{0.4} = 0.25.$$

2. An agent assigns $\mathrm{Cr}(B) = 0.75$, $\mathrm{Cr}(A \mid B) = 0.2$, $\mathrm{Cr}(A) = 0.25$. Her credences satisfy the probability axioms. What is $\mathrm{Cr}(A \mid B^c)$?

A. 0.4 **B.** 0.2

C. 0.25 **D.** 0.1

Solution. **A: 0.4.** By the complement rule, $\mathrm{Cr}(B^c) = 1 - 0.75 = 0.25$. By the law of total probability,

$$\begin{aligned}
\mathrm{Cr}(A) &= \mathrm{Cr}(A \mid B)\ \mathrm{Cr}(B) + \mathrm{Cr}(A \mid B^c)\ \mathrm{Cr}(B^c),\\
0.25 &= 0.2 \times 0.75 + 0.25\ \mathrm{Cr}(A \mid B^c),\\
0.25 &= 0.15 + 0.25\ \mathrm{Cr}(A \mid B^c),\\
0.25\ \mathrm{Cr}(A \mid B^c) &= 0.10,\\
\mathrm{Cr}(A \mid B^c) &= 0.4.
\end{aligned}$$

3. An agent with non-probabilistic credences is certain nobody will ever offer her a bet. Why does this fact not, by itself, defeat the vulnerability-based Dutch book argument for probabilism?

A. The probability axioms apply only when money will actually change hands.

B. Non-probabilistic credences guarantee that the agent will actually lose money.

C. The Dutch book theorem alone proves that every vulnerable agent is irrational.

D. The argument concerns susceptibility to a possible pattern of offers, not whether those offers actually occur.

Solution. **D.** The vulnerability-based argument criticises the credences themselves: because they violate the axioms, there is a set of bets, each of which she would accept by her own lights, that together guarantee her a loss. Being susceptible to such a pattern of offers is the defect. Whether anyone ever actually makes the offers is beside the point, so the certainty that nobody will does not, by itself, answer the argument.

4. An agent has exactly two acts, A and B, and cares only about which of X, Y results. Which statement is a dependency hypothesis?

A. If A were performed, X would result; if B were performed, Y would result.

B. If A were performed, either X or Y would result; if B were performed, Y would result.

C. If A were performed, X would result; if B were performed, Y would result; and the agent will choose A.

D. If A were performed, X would result; either A or B will be performed.

Solution. **A.** A dependency hypothesis is a maximally specific proposition about how the outcomes the agent cares about depend causally on the acts: it says, for *every* available act, which outcome would result. A does exactly that. B is not maximally specific (it leaves open what A would produce). C adds a claim about which act will be chosen, which is not part of the dependency. D says nothing about what B would produce.

5. An evidential decision theorist calculates evidential expected utility using the formula with credences conditional on the act. She describes the same decision problem using a different admissible partition into states, preserving the utilities and conditional credences of the underlying outcomes. What happens to the evidential expected utility of an act?

A. It must rise when the new partition has more states.

B. It becomes the causal expected utility.

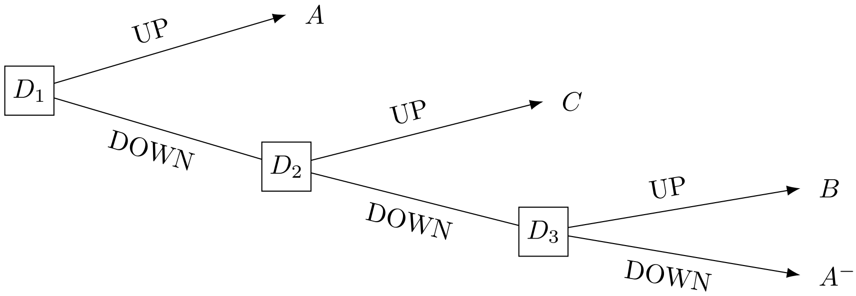
C. It remains the same only if all states are probabilistically independent of the act.

D. It remains the same.

Solution. **D.** Evidential expected utility is partition-invariant: computed with credences conditional on the act, it takes the same value on every admissible partition, because each admissible partition groups the same outcomes with the same conditional credences and utilities. No independence assumption is needed, which is what rules out C.

Questions 6–7

Preferences remain fixed: $A \succ B$; $B \succ C$; $C \succ A$; $A \succ A^-$; $A^- \succ B$; $C \succ A^-$. The agent starts with A. At every decision, UP ends the offers with the current holding; DOWN accepts a trade. A^- is A with a small fee deducted.



6. What final holding does myopic choice produce?

A. B **B.** A^-

C. A **D.** C

Solution. **B: A^-.** Myopic choice judges each trade on its own, comparing the current holding with what the trade gives. At D_1 the agent holds A and DOWN gives C; since $C \succ A$, she trades. At D_2 she holds C and DOWN gives B; since $B \succ C$, she trades. At D_3 she holds B and DOWN gives A^-; since $A^- \succ B$, she trades. The final holding is A^-.

7. What will an agent following sophisticated choice do at D_1, and why?

A. UP: $A \succ B$. **B.** DOWN: $C \succ A$.

C. UP: $A \succ A^-$. **D.** DOWN: $B \succ A$.

Solution. **B: DOWN, because $C \succ A$.** Work backwards. At D_3, UP gives B and DOWN gives A^-; since $A^- \succ B$, the agent would choose DOWN. At D_2, UP gives C and DOWN leads (via the predicted choice at D_3) to A^-; since $C \succ A^-$, the agent would choose UP. At D_1, UP gives A and DOWN leads (via the predicted choice at D_2) to C; since $C \succ A$, the agent chooses DOWN.

8. Expected utility represents the preferences, with

$$\mathrm{EU}(A) = 8, \quad \mathrm{EU}(B) = 2, \quad \mathrm{EU}(C) = -2.$$

What probability p makes $B \sim A[p]C$?

A. 0.25 **B.** 0.2

C. 0.4 **D.** 0.5

Solution. **C: 0.4.** Indifference means equal expected utility:

$$\begin{aligned}
\mathrm{EU}(B) &= p\,\mathrm{EU}(A) + (1-p)\,\mathrm{EU}(C),\\
2 &= 8p + (1-p)(-2),\\
2 &= 8p - 2 + 2p,\\
2 &= 10p - 2,\\
10p &= 4,\\
p &= 0.4.
\end{aligned}$$

9. An agent uses expected utility with $u(m) = \sqrt{m}$. Which sure monetary amount is equally preferred to

$$\$64[0.5]\$16 \quad ?$$

A. \$36.00 **B.** \$16.00

C. \$6.00 **D.** \$40.00

Solution. **A: \$36.** The expected utility of the gamble is

$$0.5\,u(64) + 0.5\,u(16) = 0.5\sqrt{64} + 0.5\sqrt{16} = 0.5 \times 8 + 0.5 \times 4 = 4 + 2 = 6.$$

A sure amount m is equally preferred exactly when $u(m) = \sqrt{m} = 6$, that is, $m = 36$. (The expected *money*, \$40, is not the answer: with $u(m) = \sqrt{m}$ the agent is risk-averse, so the sure amount she values equally is less than \$40.)

10. An agent adopts a plan. Later, she can still depart from it and would prefer to do so if she considered only the remaining options. She nevertheless follows the plan because she adopted it. Which procedure does this illustrate?

A. Physical commitment. **B.** Naive choice.

C. Myopic choice. **D.** Resolute choice.

Solution. **D.** Resolute choice carries out the plan adopted at the start even where, considering only the options that remain, the agent would now prefer to deviate. Physical commitment removes the option of deviating; a naive chooser would deviate; myopic choice does not consult a plan at all.

Section B

Answer at least three of Questions 1–4. Your best three marks will be counted. Each question is worth 20 marks.

Question 1 [20 marks]

An agent's credence function Cr is defined on all subsets of a sample space Ω. An event is a subset of Ω.

(a) State what it takes for Cr to satisfy the probability axioms, completing the following statements.

(i) **Non-negativity.** For every event X, $\mathbf{Cr}(\boldsymbol{X}) \geq \mathbf{0}$. [2]

(ii) **Normalisation.** $\mathbf{Cr}(\boldsymbol{\Omega}) = \mathbf{1}$. [2]

(iii) **Additivity.** If events X and Y are **disjoint/mutually exclusive (that is,** $\boldsymbol{X} \cap \boldsymbol{Y} = \emptyset$**)**, then

$$\mathbf{Cr}(\boldsymbol{X} \cup \boldsymbol{Y}) = \mathbf{Cr}(\boldsymbol{X}) + \mathbf{Cr}(\boldsymbol{Y}).$$

[2]

From now on, assume that the agent's credences satisfy the probability axioms.

(b) For any event A, prove that

$$\mathrm{Cr}(A^c) = 1 - \mathrm{Cr}(A).$$

Identify the probability axioms used in your proof. [4]

Q1 (b)

We have $A \cup A^c = \Omega$ and $A \cap A^c = \emptyset$, so A and A^c are disjoint. Hence, by Additivity,

$$\mathrm{Cr}(\Omega) = \mathrm{Cr}(A \cup A^c) = \mathrm{Cr}(A) + \mathrm{Cr}(A^c),$$

and by Normalisation, $\mathrm{Cr}(\Omega) = 1$. So $\mathrm{Cr}(A) + \mathrm{Cr}(A^c) = 1$. Rearranging,

$$\mathrm{Cr}(A^c) = 1 - \mathrm{Cr}(A).$$

(c) For arbitrary events X and Y, prove that if $X \subset Y$, then

$$\mathrm{Cr}(X) \leq \mathrm{Cr}(Y).$$

[4]

Q1 (c)

Since $X \subset Y$, we have $Y = (Y \setminus X) \cup X$, with $(Y \setminus X) \cap X = \emptyset$. By Additivity,

$$\mathrm{Cr}(Y) = \mathrm{Cr}\big((Y \setminus X) \cup X\big) = \mathrm{Cr}(Y \setminus X) + \mathrm{Cr}(X).$$

By Non-negativity, $\mathrm{Cr}(Y \setminus X) \geq 0$. So

$$\mathrm{Cr}(Y) = \mathrm{Cr}(Y \setminus X) + \mathrm{Cr}(X) \geq \mathrm{Cr}(X).$$

(d) For arbitrary events X and Y, prove the addition rule:

$$\mathrm{Cr}(X \cup Y) = \mathrm{Cr}(X) + \mathrm{Cr}(Y) - \mathrm{Cr}(X \cap Y).$$

Write down the disjoint decompositions you use and identify where Additivity applies. [6]

Q1 (d)

Three disjoint decompositions are used:

$$\begin{aligned} X \cup Y &= (X \setminus Y) \cup (Y \setminus X) \cup (X \cap Y), \\ X &= (X \setminus Y) \cup (X \cap Y), \\ Y &= (Y \setminus X) \cup (X \cap Y). \end{aligned}$$

In each, the events on the right are pairwise disjoint. Applying Additivity to each decomposition,

$$\begin{aligned} \mathrm{Cr}(X \cup Y) &= \mathrm{Cr}(X \setminus Y) + \mathrm{Cr}(Y \setminus X) + \mathrm{Cr}(X \cap Y), \\ \mathrm{Cr}(X) &= \mathrm{Cr}(X \setminus Y) + \mathrm{Cr}(X \cap Y), \\ \mathrm{Cr}(Y) &= \mathrm{Cr}(Y \setminus X) + \mathrm{Cr}(X \cap Y). \end{aligned}$$

Rearranging the second and third equations, $\mathrm{Cr}(X \setminus Y) = \mathrm{Cr}(X) - \mathrm{Cr}(X \cap Y)$ and $\mathrm{Cr}(Y \setminus X) = \mathrm{Cr}(Y) - \mathrm{Cr}(X \cap Y)$. Substituting these into the first,

$$\begin{aligned} \mathrm{Cr}(X \cup Y) &= \mathrm{Cr}(X) - \mathrm{Cr}(X \cap Y) + \mathrm{Cr}(Y) - \mathrm{Cr}(X \cap Y) + \mathrm{Cr}(X \cap Y) \\ &= \mathrm{Cr}(X) + \mathrm{Cr}(Y) - \mathrm{Cr}(X \cap Y). \end{aligned}$$

Alternative proof. Two disjoint decompositions suffice:

$$\begin{aligned} X \cup Y &= X \cup (Y \setminus X), \\ Y &= (X \cap Y) \cup (Y \setminus X). \end{aligned}$$

In each, the two events on the right are disjoint. Applying Additivity to each decomposition,

$$\begin{aligned} \mathrm{Cr}(X \cup Y) &= \mathrm{Cr}(X) + \mathrm{Cr}(Y \setminus X), \\ \mathrm{Cr}(Y) &= \mathrm{Cr}(X \cap Y) + \mathrm{Cr}(Y \setminus X). \end{aligned}$$

Rearranging the second equation, $\mathrm{Cr}(Y \setminus X) = \mathrm{Cr}(Y) - \mathrm{Cr}(X \cap Y)$. Substituting this into the first,

$$\mathrm{Cr}(X \cup Y) = \mathrm{Cr}(X) + \mathrm{Cr}(Y) - \mathrm{Cr}(X \cap Y).$$

The same proof works with the roles of X and Y reversed, using $X \cup Y = Y \cup (X \setminus Y)$ and $X = (X \cap Y) \cup (X \setminus Y)$.

Question 2 [20 marks]

EDT means evidential decision theory; CDT means causal decision theory. In a Newcomb-like problem, an opaque box contains either 100 tokens or none. A transparent box contains 20 tokens. An agent may take only the opaque box (O), or both boxes (B). A predictor filled the opaque box with 100 tokens if it predicted O, and left it empty if it predicted B. The boxes were filled before the choice, and the choice cannot causally affect their contents. Utility equals the number of tokens received.

Write F for the opaque box being full. The agent's credences satisfy

$$\mathrm{Cr}(F \mid O) = r, \qquad \mathrm{Cr}(F \mid B) = 1 - r, \qquad 0.5 \leq r \leq 1.$$

Thus r specifies a conditional probability for *each* act, rather than merely the predictor's overall success rate.

	F	F^c
O	100	0
B	120	20

(a) Determine $\mathrm{Cr}(F^c \mid O)$ and $\mathrm{Cr}(F^c \mid B)$ in terms of r. You may appeal to any result proved in the lectures. [2]

Q2 (a)

Credences conditional on an act satisfy the probability axioms, so the complement rule holds for them too. Hence

$$\mathrm{Cr}(F^c \mid O) = 1 - \mathrm{Cr}(F \mid O) = \mathbf{1 - \boldsymbol{r}}, \qquad \mathrm{Cr}(F^c \mid B) = 1 - \mathrm{Cr}(F \mid B) = 1 - (1-r) = \boldsymbol{r}.$$

(b) For $r = 0.8$, calculate the evidential expected utility of each act and state EDT's recommendation. [4]

Q2 (b)

Weighting each act's utilities by the credences conditional on that act, and using part (a),

$$\begin{aligned}
\mathrm{EU}_e(O) &= 100\ \mathrm{Cr}(F \mid O) + 0\ \mathrm{Cr}(F^c \mid O) = 100r + 0(1-r) = 100r = 80,\\
\mathrm{EU}_e(B) &= 120\ \mathrm{Cr}(F \mid B) + 20\ \mathrm{Cr}(F^c \mid B) = 120(1-r) + 20r\\
&= 120 - 120r + 20r = 120 - 100r = 40,
\end{aligned}$$

with $r = 0.8$. Since $80 > 40$, **EDT recommends O.**

(c) Treating r as a variable again, determine exactly when EDT strictly recommends O, when it strictly recommends B, and when it is indifferent. [8]

Q2 (c)

From part (b), $\mathrm{EU}_e(O) = 100r$ and $\mathrm{EU}_e(B) = 120 - 100r$. $\mathrm{EU}_e(O)$ increases with r and $\mathrm{EU}_e(B)$ decreases with r. So if r^* is the value at which EDT is indifferent between O and B, then EDT strictly recommends O when $r > r^*$ and strictly recommends B when $r < r^*$.

EDT is indifferent when $\mathrm{EU}_e(O) = \mathrm{EU}_e(B)$:

$$\begin{aligned} 100r^* &= 120 - 100r^*, \\ 200r^* &= 120, \\ r^* &= 0.6. \end{aligned}$$

Therefore, within the stated range $0.5 \le r \le 1$:

- **EDT strictly recommends O when** $r > 0.6$;
- **EDT strictly recommends B when** $r < 0.6$;
- **EDT is indifferent when** $r = 0.6$.

(d) Write $q = \mathrm{Cr}(F)$. Express the causal expected utility of each act in terms of q. Determine CDT's recommendation and explain why it can be determined without knowing q. [6]

Q2 (d)

The states F and F^c are causally independent of the act, so CDT weights each act's utilities by the unconditional credences q and $1 - q$:

$$\begin{aligned} \mathrm{EU}_c(O) &= 100q + 0(1-q) = 100q, \\ \mathrm{EU}_c(B) &= 120q + 20(1-q) = 100q + 20. \end{aligned}$$

Now $\mathrm{EU}_c(B) > \mathrm{EU}_c(O)$ exactly when

$$\begin{aligned} 120q + 20(1-q) &> 100q, \\ 100q + 20 &> 100q, \\ 20 &> 0, \end{aligned}$$

which holds for every q. So $\mathrm{EU}_c(B) > \mathrm{EU}_c(O)$ for all q, and **CDT recommends** B.

This can be determined without knowing q because B strongly statewise dominates O (by 20 units of utility in each state) on a partition into causally independent states.

Question 3 [20 marks]

Recall the notation:

- $a \succsim b$: a is at least as preferred as b.
- $a \succ b$: a is strictly preferred to b.
- $a \sim b$: a and b are equally preferred.

A slash through the relation denotes its negation:

- $a \not\succsim b$: a is not at least as preferred as b.
- $a \not\succ b$: a is not strictly preferred to b.
- $a \not\sim b$: a and b are not equally preferred.

(a) Write down the definitions of the following relations in terms of $\succsim$: [4]

(i) $a \succ b$.

> Q3 (a)(i)
>
> $a \succ b$ if and only if $\boldsymbol{a \succsim b}$ **and** $\boldsymbol{b \not\succsim a}$.

(ii) $a \sim b$.

> Q3 (a)(ii)
>
> $a \sim b$ if and only if $\boldsymbol{a \succsim b}$ **and** $\boldsymbol{b \succsim a}$.

(b) A relation R is **transitive** when, for all X, Y and Z, if $X\,R\,Y$ and $Y\,R\,Z$, then $X\,R\,Z$.

Suppose $\succsim$ is transitive. Using your definitions from part (a), prove that:

(i) $\succ$ is transitive. [4]

> Q3 (b)(i)
>
> Suppose $a \succ b$ and $b \succ c$. By (a)(i), $a \succsim b$ and $b \succsim c$, so by transitivity of $\succsim$, $a \succsim c$.
>
> Suppose, for a contradiction, that $c \succsim a$. Then, since $a \succsim b$, transitivity of $\succsim$ gives $c \succsim b$, contradicting that $b \succ c$ (which by (a)(i) requires $c \not\succsim b$). So $c \not\succsim a$.
>
> Hence $a \succsim c$ and $c \not\succsim a$, that is, $a \succ c$, as required.

(ii) $\sim$ is transitive. [2]

> Q3 (b)(ii)
>
> Suppose $a \sim b$ and $b \sim c$. We want to show that $a \sim c$.
>
> Since $a \sim b$, by (a)(ii) $a \succsim b$ and $b \succsim a$. Since $b \sim c$, $b \succsim c$ and $c \succsim b$. By transitivity of $\succsim$, $a \succsim b$ and $b \succsim c$ give $a \succsim c$, and $c \succsim b$ and $b \succsim a$ give $c \succsim a$. Hence $a \sim c$, as required.

(c) Consider three outcomes a, b and c. In each table below, specify a full pattern of preferences by entering a **tick** if the row outcome is at least as preferred as the column outcome, and a **cross** otherwise. The diagonal entries are already ticked, since each outcome is at least as preferred as itself.

(i) Give a pattern for which $\succ$ is transitive but $\succsim$ is **not** transitive. [2]

$\succsim$	a	b	c
a	✓	✓	×
b	✓	✓	✓
c	×	×	✓

Q3 (c)(i)

Here $a \sim b$ and $b \succ c$, while a and c are incomparable: neither $a \succsim c$ nor $c \succsim a$. Then $\succsim$ is not transitive, since $a \succsim b$ and $b \succsim c$ but $a \not\succsim c$. But $b \succ c$ is the only strict preference between distinct outcomes, so there is no chain $x \succ y$, $y \succ z$ to test, and $\succ$ is transitive.

All the patterns that work. A check by computer finds that 21 of the 64 possible tables satisfy the requirement, four up to relabelling of a, b and c. The table above is one of the four; the other three are:

$\succsim$	a	b	c
a	✓	✓	✓
b	✓	✓	✓
c	✓	×	✓

$a \sim b$, $a \sim c$, $b \succ c$

$\succsim$	a	b	c
a	✓	✓	×
b	✓	✓	×
c	✓	×	✓

$a \sim b$, $c \succ a$, $b \,|\, c$

$\succsim$	a	b	c
a	✓	✓	✓
b	✓	✓	×
c	✓	×	✓

$a \sim b$, $a \sim c$, $b \,|\, c$

($x \,|\, y$: x and y incomparable, both cells crossed.) Any of the 21 earns the marks.

(ii) Give a pattern for which $\sim$ is transitive but $\succsim$ is **not** transitive. [2]

$\succsim$	a	b	c
a	✓	✓	×
b	×	✓	✓
c	×	×	✓

Q3 (c)(ii)

In this pattern $a \succ b$ and $b \succ c$, while a and c are incomparable. Then $\succsim$ is not transitive, since $a \succsim b$ and $b \succsim c$ but $a \not\succsim c$. But no two distinct outcomes are equally preferred, so $\sim$ holds only between each outcome and itself and is (vacuously) transitive.

All the patterns that work. A check by computer finds that 26 of the 64 possible tables satisfy the requirement, five up to relabelling of a, b and c. The table above is one of the five; the other four are:

$\succsim$	a	b	c
a	✓	✓	✓
b	✓	✓	×
c	×	✓	✓

$a \sim b$, $a \succ c$, $c \succ b$

$\succsim$	a	b	c
a	✓	✓	✓
b	✓	✓	×
c	×	×	✓

$a \sim b$, $a \succ c$, $b \,|\, c$

$\succsim$	a	b	c
a	✓	✓	×
b	✓	✓	×
c	✓	×	✓

$a \sim b$, $c \succ a$, $b \,|\, c$

$\succsim$	a	b	c
a	✓	✓	×
b	×	✓	✓
c	✓	×	✓

$a \succ b$, $b \succ c$, $c \succ a$

Any of the 26 earns the marks.

(d) Suppose that $\succ$ and $\sim$ are both transitive, and that $\succsim$ is complete. Complete the following proof that $\succsim$ is transitive, filling each gap with one comparison or the name of a property. [6]

Proof. Suppose $a \succsim b$ and $b \succsim c$. We must show that $\underline{\boldsymbol{a \succsim c}}$. By the definitions in part (a), there are four cases.

Case 1: $a \sim b$ **and** $b \sim c$**.**

By transitivity of $\sim$, $\underline{\boldsymbol{a \sim c}}$. Hence $a \succsim c$.

Case 2: $a \succ b$ **and** $b \sim c$**.**

By **completeness**, either $a \sim c$, $c \succ a$, or $a \succ c$.

- If $a \sim c$, then since $c \sim b$ (as this is equivalent to $b \sim c$), transitivity of $\sim$ gives $\underline{\boldsymbol{a \sim b}}$. This contradicts that $\underline{\boldsymbol{a \succ b}}$.
- If $c \succ a$, transitivity of $\succ$ gives $\underline{\boldsymbol{c \succ b}}$. This contradicts that $\underline{\boldsymbol{b \sim c}}$.
- Therefore $a \succ c$, and hence $a \succsim c$.

Case 3: $a \sim b$ **and** $b \succ c$**.**

Again, by **completeness**, either $a \sim c$, $c \succ a$, or $a \succ c$.

- If $a \sim c$, then since $b \sim a$ (as this is equivalent to $a \sim b$), transitivity of $\sim$ gives $\underline{\boldsymbol{b \sim c}}$. This contradicts that $\underline{\boldsymbol{b \succ c}}$.
- If $c \succ a$, transitivity of $\succ$ gives $\underline{\boldsymbol{b \succ a}}$. This contradicts that $\underline{\boldsymbol{a \sim b}}$.
- Therefore $\underline{\boldsymbol{a \succ c}}$, and hence $a \succsim c$.

Case 4: $a \succ b$ **and** $b \succ c$**.**

By transitivity of $\underline{\succ}$, we have $\underline{\boldsymbol{a \succ c}}$. Hence $a \succsim c$.

In every case, $a \succsim c$. Therefore $\succsim$ is transitive. $\square$

Question 4 [20 marks]

Recall the notation:

- $a \succsim b$: a is at least as preferred as b.
- $a \succ b$: a is strictly preferred to b.
- $a \sim b$: a and b are equally preferred.

A slash through the relation denotes its negation:

- $a \not\succsim b$: a is not at least as preferred as b.
- $a \not\succ b$: a is not strictly preferred to b.
- $a \nsim b$: a and b are not equally preferred.

Suppose preferences satisfy Independence: for any prospects X, Y, Z and $p \in (0, 1)$,

$$X \succsim Y \quad \text{if and only if} \quad X\,[p]\,Z \succsim Y\,[p]\,Z \quad \text{if and only if} \quad Z\,[p]\,X \succsim Z\,[p]\,Y.$$

Here $X\,[p]\,Y$ gives prospect X with probability p, and prospect Y otherwise.

(a) Complete the following proof of the forward direction of strict Independence, filling each gap with one comparison or the name of a property. [10]

$$A \succ B \quad \text{if and only if} \quad A\,[p]\,C \succ B\,[p]\,C \qquad (p \in (0, 1)).$$

Suppose $A \succ B$. By the definition of strict preference,

$$A \succsim B \quad \text{and} \quad \underline{\boldsymbol{B \not\succsim A}}.$$

By **Independence**, it follows that $A\,[p]\,C \succsim B\,[p]\,C$.

Suppose, for a contradiction, that $B\,[p]\,C \succsim A\,[p]\,C$.

By Independence, this gives $\underline{\boldsymbol{B \succsim A}}$, contradicting that $\underline{\boldsymbol{B \not\succsim A}}$.

Therefore $\underline{\boldsymbol{B\,[p]\,C \not\succsim A\,[p]\,C}}$. Together with $A\,[p]\,C \succsim B\,[p]\,C$, this establishes $A\,[p]\,C \succ B\,[p]\,C$.

Also accepted. After "contradicting that", $A \succ B$. And $A \nsim B$ wherever $B \not\succsim A$ appears: given $A \succsim B$, the two are equivalent.

Question 4 (continued)

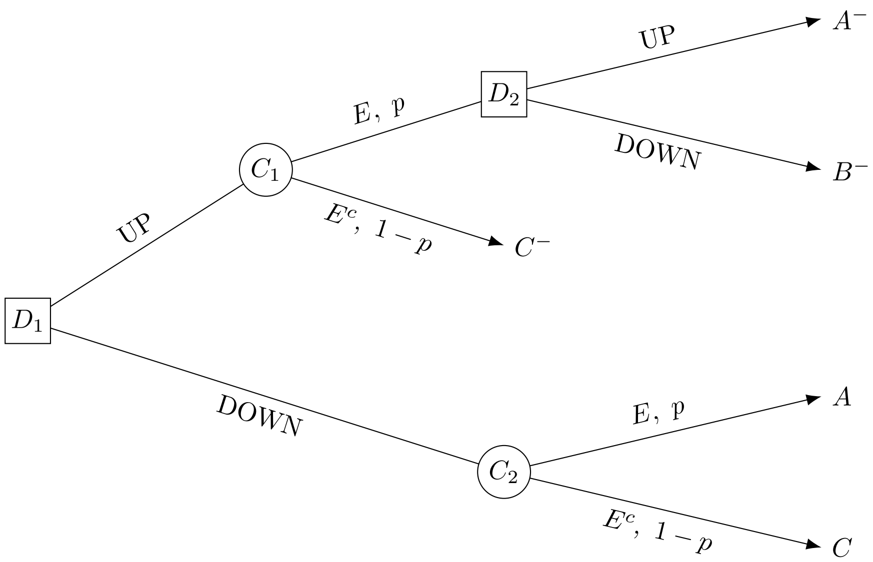
An agent has the following preferences:

$$A \succ B, \qquad B[p]C \succ A[p]C, \qquad 0 < p < 1.$$

Consider the tree below. Both chance nodes resolve the same event E, with probability p. Write X^- for X with the cost $c > 0$ deducted. The agent knows the whole tree and has these additional preferences, held fixed throughout:

$$A^- \succ B^-, \qquad B^-[p]C^- \succ A[p]C, \qquad A[p]C \succ A^-[p]C^-,$$

$$B^-[p]C^- \succ A^-[p]C^-.$$



Squares are decision nodes; circles are chance nodes. At D_1, UP pays c and trades; DOWN refuses. At D_2, UP exchanges for A; DOWN keeps B.

(b) Suppose that the agent follows sophisticated choice: work backwards, predicting later choices and comparing the resulting prospects.

(i) State their choice at each decision node if it is reached. [6]

> Q4 (b)(i)
>
> **At D_2:** $A^- \succ B^-$, so the agent goes **UP**.
>
> **At D_1:** UP leads to D_2, where (as just predicted) the agent goes UP, so the prospect from UP is $A^-[p]C^-$. DOWN involves no further choices, so the prospect from DOWN is $A[p]C$. Since $A[p]C \succ A^-[p]C^-$, the agent goes **DOWN**.
>
> So the full plan is (D_1: DOWN, D_2: UP), though the agent never reaches D_2.

(ii) Give the prospect the agent obtains, evaluated from D_1, by following sophisticated choice. [4]

> Q4 (b)(ii)
>
> From (b)(i), the agent goes DOWN at D_1, and there are no further choices, so the prospect they obtain is $\boldsymbol{A[p]C}$.