PE3101P: Decision Theory and Social Choice

In-class Test 2 Practice Paper: Section B

Answer at least three of the four questions. Your best three marks will be counted. Each question is worth 20 marks.

Question 1 [20 marks]

(a) Write πX for the outcome obtained from X by a rearrangement π of who gets which well-being level. A ranking $\succsim$ of outcomes for three people satisfies Permutation Invariance and Transitivity, recalled below. Suppose that

$$(4, 1, 1) \succ (1, 1, 2).$$

Nothing else is known about the ranking.

For each pair of outcomes in (i) and (ii), there are four possibilities: the first is better, the second is better, they are equally good, or they are incomparable. Which of these does the information rule out? Put a cross in the table under each possibility that is ruled out. Whenever you use Permutation Invariance or the fact recalled in (ii), say which rearrangement you apply, by saying who gets whose level (for example: "persons 2 and 3 swap levels"). You do not need to show that the possibilities you leave uncrossed can actually occur.

(i) Recall the principle of

Permutation Invariance: For any rearrangement π: $X \succsim Y$ iff $\pi X \succsim \pi Y$.

The pair is $(1, 4, 1)$ and $(2, 1, 1)$. [2]

Pair	First better	Second better	Equally good	Incomparable
$(1, 4, 1)$ and $(2, 1, 1)$				

(ii) Recall the principle of

Transitivity: If $X \succsim Y$ and $Y \succsim Z$, then $X \succsim Z$. (So also: if $X \succsim Y$ and $Y \succ Z$, then $X \succ Z$.)

Recall also that, given Permutation Invariance and Transitivity, no rearrangement of an outcome is better than it: $\pi X \succ X$ is impossible, for any outcome X and any rearrangement π. The pair is $(4, 1, 1)$ and $(2, 1, 1)$. [2]

Pair	First better	Second better	Equally good	Incomparable
$(4, 1, 1)$ and $(2, 1, 1)$				

(b) In this part, w_i is person i's wellbeing, and everyone's wellbeing is positive. Recall that Gini egalitarianism ranks outcomes by total wellbeing $\times$ (1 $-$ Gini coefficient), the outcome's Gini value, and that this can be written as

$$\text{Gini value} = \text{total wellbeing} - \frac{S}{2n},$$

where n is the number of people and S is the sum of the gaps $|w_i - w_j|$ over all i and j (each pair of different people counted both ways round). A pure transfer of wellbeing means

that one person loses some wellbeing, another gains the same amount, and everyone else is unaffected. Recall the principle of

> **Pigou–Dalton**: A pure transfer of wellbeing from someone better off to someone worse off is an improvement, as long as the person who gave is still at least as well off as the person who received.

In (i)–(iii) we prove that Gini egalitarianism satisfies Pigou–Dalton, by seeing how the transfer changes the terms of S. Take any outcome, and any pure transfer of wellbeing in it that meets the conditions in Pigou–Dalton. The giver starts at level w_g and the receiver at level w_r, with $w_r < w_g$. The giver gives t units, where $t > 0$. So afterwards the giver has $w_g - t$ and the receiver has $w_r + t$, with $w_r + t \leq w_g - t$.

(i) The sum S contains two terms for the gap between these two people, $|w_g - w_r|$ and $|w_r - w_g|$. By how much does the pure transfer change the sum of these two terms? [3]

Now take any third person, at level w_k, who is unaffected by the pure transfer. Think of $|w_g - w_k|$ and $|w_r - w_k|$ as the distances, on a number line, from this person to the giver and to the receiver. If $w_k \leq w_r$ or $w_k \geq w_g$, the transfer moves one of these two distances down by t and the other up by t, so their sum is unchanged.

(ii) Show that if $w_r < w_k < w_g$, the pure transfer does not make the sum of the two distances larger. Consider separately the cases where w_k is below $w_r + t$, from $w_r + t$ to $w_g - t$, and above $w_g - t$. [7]

(iii) Use (i), (ii) and the cases $w_k \leq w_r$ and $w_k \geq w_g$ above to show that the pure transfer raises the Gini value. [3]

(iv) The proof used the condition in Pigou–Dalton that the giver is still at least as well off as the receiver. Show that it is needed: give an outcome for two people, and a pure transfer of wellbeing in it from the better off person to the worse off person, that lowers the Gini value. [3]

Question 2 [20 marks]

Recall that an *option set* S is a set of available options, and a *choice function* C gives, for every option set S, a non-empty subset $C(S)$ of S: what may be chosen from S. From an option set with only one option, that option is chosen.

(a) Melissa likes her coffee sweet, but she can only taste a difference in sweetness of at least two spoonfuls of sugar. Three cups of coffee, a, b and c, have 0, 1 and 2 spoonfuls of sugar respectively. From each option set, Melissa may choose any cup unless another cup in the option set has at least two more spoonfuls than it. Let C be her choice function.

(i) Fill in the table. [4]

Option set	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
C(option set)				

Recall the following conditions.

Contraction consistency (Property α): If x is chosen from an option set S, and x is in a smaller option set T inside S, then x is chosen from T.

$$[x \in C(S) \text{ and } x \in T \subset S] \;\Rightarrow\; x \in C(T)$$

Expansion consistency (Property γ): If x is chosen from an option set S_1, and x is chosen from an option set S_2, then x is chosen from the option set that combines them.

$$[x \in C(S_1) \text{ and } x \in C(S_2)] \;\Rightarrow\; x \in C(S_1 \cup S_2)$$

Property β: If x and y are both chosen from T, and $T \subset S$, then y is chosen from S if and only if x is.

$$[x, y \in C(T) \text{ and } T \subset S] \;\Rightarrow\; [y \in C(S) \Leftrightarrow x \in C(S)]$$

For each of (ii)–(iv): does C satisfy the condition? If it does, say why. If it does not, give option sets and options that show this.

(ii) Contraction consistency (Property α). [2]

(iii) Expansion consistency (Property γ). [2]

(iv) Property β. [2]

(b) Melissa is also choosing a flat. She compares any two flats by their rent, their size and the time it takes to get to work from them, and from any two flats she chooses the one that is better on at least two of these three. For three flats, x, y and z, this gives

$$C(\{x, y\}) = \{x\}, \qquad C(\{y, z\}) = \{y\}, \qquad C(\{z, x\}) = \{z\}.$$

We are not told what she chooses from all three flats. Show that, whatever it is, Melissa's choice function C violates contraction consistency (Property α). Recall that $C(S)$ is never empty. [4]

(c) Now take any number $n \geq 3$ of different options, $x_1, x_2, \ldots, x_n$, and write x_{n+1} for x_1. Let C be a choice function on their option sets such that, for each i from 1 to n,

$$C(\{x_i, x_{i+1}\}) = \{x_i\}.$$

Show that C violates contraction consistency (Property α). [6]

Question 3 [20 marks]

A social ordering function gives a social ordering $\succsim$, complete and transitive, for each profile of voters' orderings. In this question there are finitely many voters and at least three options, one of which is b, and the social ordering function satisfies

Unrestricted Domain: Every profile is allowed: voters may hold any orderings at all.

Pareto: If every voter prefers x to y, then $x \succ y$.

IIA: Take any two profiles, and any two options x and y. If every voter ranks x against y the same way in both profiles, then society ranks x against y the same way in both.

This question is about the following lemma, which follows from these three conditions.

Extreme Option Lemma: For any social ordering function satisfying Unrestricted Domain, Pareto and IIA: if each voter ranks b either strictly above all other options or strictly below all of them (different voters may differ in which), then society ranks b strictly above all other options, or strictly below all of them.

In (a) you may use the lemma.

(a) In this part there are three voters and three options, a, b and c, and R is the profile below.

R:

Voter	Ranking
1	$b \succ a \succ c$
2	$a \succ c \succ b$
3	$c \succ a \succ b$

(i) Is it possible that society ranks $a \sim b$ at R? Say why or why not. [1]

(ii) Suppose that society ranks $a \succ b$ at R. How does society rank b against c at R? Say why. [2]

(iii) The profile T below is the same as R, except that voter 2's ordering is now $a \succ b \succ c$.

T:

Voter	Ranking
1	$b \succ a \succ c$
2	$a \succ b \succ c$
3	$c \succ a \succ b$

Suppose that society ranks $a \succ b$ at T. Must society rank $c \succ b$ at T? If so, prove it. If not, say why the answer to (ii) does not carry over to T, and give a social ordering function that satisfies Unrestricted Domain, Pareto and IIA and, at T, ranks $a \succ b$ but not $c \succ b$. [5]

(b) In this part we prove the lemma, so you must not use it here. Take any profile P in which each voter ranks b either strictly above all other options or strictly below all of them. We show that at P, society ranks b strictly above all other options or strictly below all of them. Suppose, for contradiction, that it does neither.

Since society does not rank b strictly above all other options, there is an option a, other than b, such that society ranks $a \succsim b$. Since society does not rank b strictly below all other

options, there is an option c, other than b, such that society ranks $b \succsim c$. It may be that $a = c$, so that society ranks $a \sim b$. In that case, let d be any option other than a and b.

(i) Show that, in that case, there are still two different options other than b: one that society ranks at least as high as b, and one that it ranks no higher than b. [1]

So, renaming if necessary, we may take a and c to be different options, with $a \succsim b$ and $b \succsim c$ at P.

To see how the proof goes on, take an example with three options, a, b and c, and four voters, in which P is the profile below. Each voter ranks b strictly above both other options or strictly below both. Suppose, as above, that society ranks $a \succsim b$ and $b \succsim c$ at P.

P:

Voter	Ranking
1	$b \succ a \succ c$
2	$a \sim c \succ b$
3	$a \succ c \succ b$
4	$b \succ a \sim c$

(ii) Write down a profile P', made from P by changing only where each voter puts c, such that IIA carries society's rankings of a against b and of b against c over from P to P', and Pareto gives $c \succ a$ at P'. [4]

P':

Voter	Ranking
1	
2	
3	
4	

(iii) Show that, under this supposition, society's ranking at P' would not be transitive, so that the supposition is false. Name each condition you use. (Hint: assume that society's ranking at P' is transitive, and derive a contradiction.) [5]

(iv) Now take any profile, with any number of options, in which each voter ranks b either strictly above all other options or strictly below all of them. Say where to move c in each voter's ordering so that the new profile meets the same requirements as P' in (ii), and why this is always possible. This is why the proof works for every such profile. [2]

Question 4 [20 marks]

There are two people, 1 and 2, and nobody else. Each ranks the social states from best to worst, with no ties. Recall:

> **Pareto**: If everyone strictly prefers s to t, then t is not chosen from any option set containing s.
>
> **Decisiveness**: A person is decisive over a pair of social states $\{s, t\}$ when, whichever of the two they prefer, the other is not chosen from any option set containing both.

Say that a state in an option set is *excluded* when, by Pareto or by someone's decisiveness, it is not chosen from that option set. In each part, the people are decisive over the pairs given there, and over no other pairs.

(a) Person 1 is decisive over one pair of social states, and person 2 over another. Which of Pareto, person 1's decisiveness and person 2's decisiveness can ever exclude person 1's favourite state (the one person 1 ranks highest) from an option set? [1]

For each of (b) and (c): is there a ranking for each person that fits the description, and under which every state in the option set named is excluded? If so, give one, and for each state say what excludes it, and through which other state. If not, say why not.

(b) There are four social states, w, x, y and z. Person 1 is decisive over $\{w, y\}$ and prefers w to y. Person 2 is decisive over $\{x, z\}$ and prefers x to z. The option set is $\{w, x, y, z\}$. [7]

	Person 1	Person 2
Best		
Second		
Third		
Worst		

(c) There are three social states, x, y and z. Person 1 is decisive over $\{x, z\}$ and prefers x to z. Person 2 is decisive over $\{y, z\}$ and prefers z to y. The option set is $\{x, y, z\}$. [4]

	Person 1	Person 2
Best		
Middle		
Worst		

(d) There are four social states, w, x, y and z. Person 1 is decisive over $\{x, z\}$. Person 2 is decisive over $\{y, z\}$ and also over $\{w, y\}$. No preferences are given. Is there a ranking for each person under which every state in $\{w, x, y, z\}$ is excluded from $\{w, x, y, z\}$, but not every state in $\{x, y, z\}$ is excluded from $\{x, y, z\}$? If so, give one, and show that it has both properties. If not, say why not. [8]

	Person 1	Person 2
Best		
Second		
Third		
Worst		