Sets and Mathematical Proofs: A Primer

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Pre-preamble: Is it worth your time to read this primer?

If you have taken rigorous mathematics courses in university, you likely know everything here already. Please briefly skim the primer to confirm.

If you have not taken any significant amount of mathematics at university yet, this primer is for you.

Contents

1 Introduction

This is a quick primer intended to introduce philosophy students to a few basic ways of manipulating sets. It also serves as an introduction to constructing mathematical proofs in general.

Mathematics at university is a bit different to what you are likely used to from high school. There is less of an emphasis on calculation and much more of an emphasis on understanding precise definitions and producing rigorous proofs.

The idea of constructing a proof might sound intimidating. Don't worry—it isn't. A proof is just a persuasive argument without gaps (or where it is easy to see how the gaps can be completely filled in). We generally proceed one step at a time, and each step can be one of two things:

(i) An **assumption** that we have made, explicitly or implicitly.

(ii) A **logical deduction** we can make from the previous steps.

An assumption will generally be something you have been explicitly told, or something that is so obvious that nobody could reasonably doubt it. Exactly what counts as "sufficiently obvious" depends on the context. For example, you might appeal to the fact that between any distinct real numbers x and z, there is a number y which is strictly between them. In any philosophy course, this is obvious enough to need no further argument. In an analysis course in a mathematics department, it might be something that needs its own proof (for example, by letting $y = \frac{x+z}{2}$ and showing that this number is in between x and z).

A logical deduction sounds like a complicated thing, but as you will see, it's actually usually a very simple step. For example, I might have a particular shape, and I've already shown that it must be either a triangle or a square. If I can then also show that the shape is not a triangle, then I can conclude that it must be a square. Logical deductions aren't scary, they're usually simple, obvious steps like this.

Sometimes, students worry that they won't be able to do mathematics. But the truth is that **anyone can do mathematics** with a bit of patience and hard work. That's not to say that everyone can do really *difficult* mathematics, the kind that research mathematicians do. I can't do that kind of thing myself. But anyone who can reason logically can do the basics, and yes, that includes **you**, the reader.

I'm not teaching you a mathematics course. But I am (most likely) teaching you a philosophy course which involves some proofs or formal results. This primer is designed to get you up to speed with some of the basics which will be useful for understanding these formal results. Specifically, we're going to learn what sets are and how to manipulate them. The reason we're learning this is that manipulating sets is essential to many formal results across philosophy and economics (and many other areas besides).

Along the way, I will give you some examples of mathematical proofs so you can see how they work. There are also exercises which ask you to complete a few proofs yourself. Please attempt the exercises. The best way to learn anything in mathematics thoroughly is to do it yourself. Learning mathematics without proving things yourself is like learning to eat without bringing the food to your mouth with your own hands. It's suitable for children, but you'll never eat independently that way.

2 What is a set?

A set is a collection of objects. They could be any objects. Numbers, bowls, letters, football teams—you name it. A set *is* a collection in the sense that there is nothing to a set beyond the items—we call them the "elements"—it contains. That means that you can't have two sets with the very same elements: a set *is* just the collection of elements, so if you've got the same collection, it's the same set, not two sets.

Saying that there are two distinct sets which contain exactly the same objects is like saying that there are two distinct families which contain exactly the same people. To mean that literally is to misunderstand what a family *is* and what makes different families different: families are different because they have different members, just like sets are different when they have different members.

3 The axiom of extensionality

This, then, is arguably the most fundamental fact about sets: they are fully specified by their elements in the sense that if two sets have exactly the same elements, then they are not two sets but one set. (That is, those two sets are numerically identical.)

This is known as the axiom of extensionality:

Axiom 1 (Extensionality)**.** If two sets have exactly the same elements, then they are identical.

Mathematicians, because they like scary notation, sometimes write it down like this:[1]

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y)$$

But what this *means* is just what I said in English: if two sets have exactly the same elements, they are identical. The lesson here, by the way, is that mathematics often *looks* much scarier than it really *is*.

Notice, by the way, that I've been talking a lot about "sets" and "belonging to" and "membership", but I haven't given you a clean definition of what these things *are*. That's not an accident, because these things don't really have a full, rigorous definition in the usual sense. And there's a good reason for that: a definition is generally a kind of reduction of advanced concepts to simpler concepts. We might, for example, define a bachelor as an unmarried man; this is a definition, but what it does is define one complex concept, that of a bachelor, in terms of two simpler concepts, that of a man and an unmarried person. Definitions in mathematics work exactly the same way: we define one thing *in terms of* other things. But we cannot define *all* of our concepts like this, else our definitions would be circular. We have to bottom out somewhere, and the concepts of "set" and "membership" actually turn out to be plausible candidates for where *all of mathematics* might bottom out.[2]

What can be said is that the axiom of extensionality does a lot to *tacitly* define what sets are. Sets are the kinds of things which are defined by their elements: once you have fully specified which elements belong to a set, you have fully specified the set. Philosophers of mathematics call this the idea of implicit definition. (There are other axioms of set theory which put further flesh on the bones, but we won't need to concern ourselves with those axioms in this primer.)

4 Sets within sets

A general point I should also make about sets before we go further is that in general it **does** make sense to ask if a set is itself an element of a further set. While sets are fully defined by and constituted by their elements, this does not mean that "a set is nothing more than its elements". From a logical perspective, the set itself counts as a further object. Thus, for example, there is a set X containing precisely the following elements:

- The number 7
- The number 8

And there is also a set Y containing precisely the following elements:

- The number 7
- The number 8
- The set X

[1] This is written in the language of set theory, where the quantifiers are tacitly taken to range over only the pure sets. If you don't know what that means, don't worry about it.

[2] A logician might put it like this: the belonging to symbol, which by the way is $\in$, is a primitive predicate—indeed, the only non-logical primitive predicate—in the language of set theory.

Set Y is then a set containing *three* elements, and it is not the same set as set X (because set X contains only two elements).

Sets can contain *anything*, including other sets. This feels strange at first, but you'll soon get used to it.

5 Roster notation

The simplest way to write down a set is to write down a list of its elements. To denote that we're writing a set, we enclose the elements in curly brackets, '{' and '}':

Definition 1 (Roster notation)**.** $\{a_1, \ldots, a_n\}$ denotes the set whose elements are exactly $a_1, \ldots, a_n$.

For example, the notation $\{7, 8, 9, 10\}$ denotes the set which contains the numbers $7, 8, 9$ and 10, and nothing else.

The order in which the elements are written down doesn't matter. Thus, $\{7, 8, 9\} = \{8, 9, 7\}$. This is because, for a set, the relevant elementhood facts are just: *is this object an element of the set or not*? Membership doesn't come in any particular order, and an object can't be an element of a set twice. It just is, or isn't, an element, and there's nothing further to say.

By convention, if we use roster notation while writing a given element multiple times, we simply assume that the element is a member of the set and treat it as though we had only written it once. Thus $\{7, 8\} = \{7, 7, 7, 7, 8, 8, 7, 8, 8, 7, 8\}$.

6 Inclusion criterion notation

The other common way to write down a set is to give a condition which determines whether or not any given element is a member of that set. This generally involves writing a variable (often x), followed by a colon, followed by the condition that the variable needs to satisfy in order to be a member of the set. (Some authors prefer to write a vertical line, '|', in place of the colon.)

For example, we might write down $\{x : x \text{ is an even number}\}$, which denotes the set which contains the even numbers and nothing else. This is particularly useful when we need to write down sets with infinitely many members, which is more common than you might think.

The condition itself—what an object has to satisfy in order to be an element of the set—is what I will call the *inclusion criterion* for that set:

Definition 2 (Inclusion criterion)**.** Where C is a condition, $\{x : C\}$ denotes the set whose elements are exactly the objects satisfying C, and C is the *inclusion criterion* for that set.

So the inclusion criterion for $\{x : x \text{ is an even number}\}$ is that x is an even number.

7 What sets are there?

You might wonder which sets, exactly, exist. The answer to this question is somewhere between "complicated" and "unknown". But for the kinds of sets we will be concerned with, it will be fine for us to tacitly apply what is sometimes called the *naive comprehension* principle:

Axiom 2 (Naive comprehension)**.** For any inclusion criterion whatsoever, there is a set whose elements are precisely the objects satisfying it.

You might have heard that Bertrand Russell showed that this principle (or rather, a closely related principle due to Gottlob Frege, known as Basic Law V) is inconsistent. That is true, the axiom of naive comprehension *is* inconsistent, and this is exactly why the question of which sets exist gets complicated.

Mathematicians have come up with sets of axioms which give us rules for when sets exist, the most famous of which are called the Zermelo–Fraenkel axioms. Unlike the naive comprehension principle, these axioms are probably consistent, but they are complicated, and they need not bother us. For our purposes, it will be fine to act as though the naive comprehension principle is true, and that any sets we can think of really do exist.

8 Subsets

A very important concept for dealing with sets is the concept of a *subset*. We say that X is a subset of Y when Y extends X. That is, every element of X is also an element of Y. Note that we do *not* require that Y contains more elements than X: if X and Y are identical, then every element of X is an element of Y, and so X is a subset of Y (that is, every set is a subset of itself):

Definition 3 (Subset)**.** $X \subset Y$ means that every element of X is an element of Y. We then say that X is a *subset* of Y.

We write $X \subset Y$ to denote that X is a subset of Y.[3] This is just because using symbols is often faster and easier than using English—but in principle, everything could be done in English.

It often helps to picture a subset as sitting inside its superset.

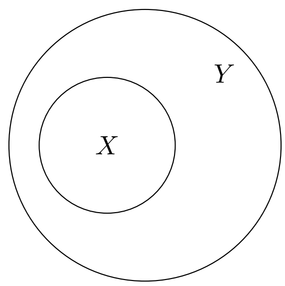

Figure 1: A subset X of a set Y.

We will often want to show that two sets X and Y are identical. An extremely important proof strategy here is to show that X is a subset of Y and Y is also a subset of X. If we can show this, we have shown that the two sets are the same. Let me explain why; and this will also serve as the first proof in this primer.

Proposition 1 (double inclusion)**.** *For any sets X and Y, if $X \subset Y$ and $Y \subset X$, then $X = Y$.*

Proof. Let X and Y be any two sets, and suppose that $X \subset Y$ and $Y \subset X$. In order to show that $X = Y$, we will show that they have exactly the same elements, and then apply the Axiom of Extensionality. So let x be any object. If it is an element of X, then since $X \subset Y$ we can conclude that x is also an element of Y. Similarly, if x is an element of Y, then since $Y \subset X$ we can conclude that x is also an element of X. Thus x is an element of X if and only if x is an element of Y.

Since x was chosen arbitrarily, this holds of any object whatsoever. In other words, any object whatsoever is either an element of both X and Y, or it is an element of neither; and this is what it means to say that X and Y have exactly the same elements. The Axiom of Extensionality then tells us that X and Y are identical. $\square$

Let me explain in more detail how that proof worked.

The statement we wanted to prove, Proposition 1 (double inclusion), is a universal statement that holds for any sets X and Y. (That is, it begins with a string of two universal quantifiers, which we rendered in English.) The way to prove a universal statement is to introduce X and Y as generic sets, *assume nothing*

[3] Some authors use $\subset$ only for *proper* subsets. X is a proper subset of Y when every element of X is an element of Y, but X and Y are not identical. Those authors write $X \subseteq Y$ for a subset in the sense defined here. The symbol $\subseteq$ always means a subset in this inclusive sense, so it is never ambiguous.

whatsoever about them, and then prove that the statement holds of X and Y. If we can do so without making any special assumptions about X and Y, then we have proved that that statement is true for any two sets whatsoever. After all, for any two sets S_1 and S_2, we could take whatever argument we made regarding X and Y and apply it directly to S_1 and S_2. (The reason we know we can do this is that we made no special assumptions about X and Y, so there is no reason why our argument should not apply to *any* two sets.)

This kind of logical inference, deriving a universal statement from what we can show of an arbitrarily chosen object about which we have made no special assumptions, is known as **universal generalisation**.

But what *is* the statement we wished to prove about the arbitrarily chosen sets X and Y? It is that, if these two sets are subsets of each other, then they are identical. This is a conditional statement. The way to prove a conditional statement is to assume the part in front—the **antecedent**—which in this case is that $X \subset Y$ and $Y \subset X$. We then attempt to show the part at the end—the **consequent**—which in this case is that $X = Y$.

So, having chosen X and Y arbitrarily, what we did was to assume the antecedent, that $X \subset Y$ and $Y \subset X$, and derive the consequent, that $X = Y$. Doing this completed the proof of the universal statement as a whole. And that is why the proof works.

I deliberately wrote this proof in a way that was longer and more detailed than we would usually require. This is because I wanted to go slowly and be clear about exactly how each piece of it worked. But normally we will want to write proofs more concisely. Here is an alternative, more concise proof of Proposition 1 (double inclusion):

Proof. Suppose that $X \subset Y$ and $Y \subset X$. For any object x, since $X \subset Y$ it holds that if $x \in X$, then $x \in Y$. Similarly, it holds that if $x \in Y$, then $x \in X$. Hence $x \in X$ if and only if $x \in Y$, and therefore by the Axiom of Extensionality we have $X = Y$. □

Now that you have seen how proofs like these work, give this one a try:

Proposition 2 (every set is a subset of itself)**.** *Every set is a subset of itself.*

Exercise 1. Prove Proposition 2 (every set is a subset of itself).

9 More about subsets

If X is a subset of Y, we also say that Y is a *superset* of X, and we write $Y \supset X$. This is the same fact written the other way around, just as "x is less than y" and "y is greater than x" say the same thing:

Definition 4 (Superset)**.** $Y \supset X$ means that $X \subset Y$. We then say that Y is a *superset* of X.

We have already noticed that every set is a subset of itself. Subsets also chain together, which is the next thing to prove.

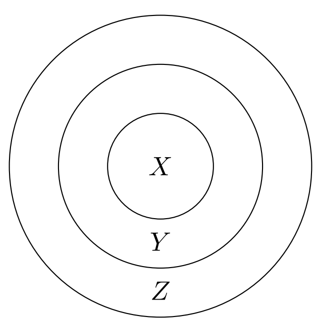

Figure 2: Proposition 3 (transitivity of inclusion): subsets chain together.

Proposition 3 (transitivity of inclusion). *For any sets X, Y and Z, if $X \subset Y$ and $Y \subset Z$, then $X \subset Z$.*

Proof. Let x be an arbitrary element of X. Since $X \subset Y$, we have $x \in Y$. Since $Y \subset Z$, we have $x \in Z$. Hence every element of X is an element of Z. $\square$

Sometimes we want to say that X is a subset of Y and that the two sets are *not* identical. In that case we say that X is a *proper* subset of Y, and we write $X \subsetneq Y$:

Definition 5 (Proper subset). $X \subsetneq Y$ means that $X \subset Y$ and $X \neq Y$.

We will hardly ever need this.

10 The empty set

There is a set with no elements whatsoever. We call it the *empty set*, and we denote it by $\varnothing$. Using the list notation from above, we could equally write it $\{\}$:

Definition 6 (Empty set). $\varnothing$ is the set with no elements at all.

The first thing to prove about such a set usually takes a moment to accept.

Proposition 4 (an empty set is a subset of anything). *For any sets X and Y, if X contains no elements, then $X \subset Y$.*

Proof. If X contains no elements, then every element of X is an element of Y, because all of the zero elements which belong to X satisfy this property. $\square$

More generally, a claim of the form "every A is a B", where there are no As at all, is always counted as true. Mathematicians say that such a claim is **vacuously** true.

Sometimes a statement is very easy to prove once we already have some other statement in hand. When that happens, we call the easy statement a **corollary** of the one we are leaning on. The next proposition is a corollary of Propositions 1 (double inclusion) and 4 (an empty set is a subset of anything).

Proposition 5 (the empty set is unique). *Any two sets with no elements are identical.*

Exercise 2. Prove Proposition 5 (the empty set is unique), as a corollary of Propositions 1 (double inclusion) and 4 (an empty set is a subset of anything).

This is what justifies us in speaking of *the* empty set rather than *an* empty set.

The empty set has nothing inside it, so there is nothing for a subset of it to contain either.

Proposition 6 (the only subset of $\varnothing$). *For any set X, if $X \subset \varnothing$, then $X = \varnothing$.*

Proof. Suppose for contradiction that there is some $x \in X$. Then since $X \subset \varnothing$, we have $x \in \varnothing$, contradicting the definition of $\varnothing$. Hence there is no $x \in X$ and by Proposition 5 (the empty set is unique) it follows that $X = \varnothing$. $\square$

One warning while we are here. Do not confuse $\varnothing$ with $\{\varnothing\}$. The first has no elements at all. The second is a set which contains exactly one element, namely the empty set. So, by the Axiom of Extensionality, they are not the same set.

11 Equivalent inclusion criteria

Before we go on to the operations on sets, here is a second general method for showing that two sets are identical. Call two conditions *equivalent* when every object satisfying one of them also satisfies the other:

Definition 7 (Equivalent conditions)**.** Two conditions are *equivalent* when every object satisfying either one of them also satisfies the other.

Proposition 7 (equivalent criteria give the same set)**.** *If two sets have equivalent inclusion criteria, then they are identical.*

Proof. Let X and Y be sets whose inclusion criteria are equivalent, and let x be any object. Then $x \in X$ if and only if x satisfies the inclusion criterion for X, and likewise $x \in Y$ if and only if x satisfies the inclusion criterion for Y. Since the two criteria are equivalent, x satisfies the one if and only if it satisfies the other. So $x \in X$ if and only if $x \in Y$. As this holds for any object x, the two sets have exactly the same elements, and so they are identical by the Axiom of Extensionality. □

The method this gives us is as follows: to show that two sets are identical, write down the inclusion criteria for each of them, and show that the two criteria are equivalent. We will not cite Proposition 7 (equivalent criteria give the same set) when we use the method, but it is what justifies it.

12 Intersections

Given two sets X and Y, their *intersection* is the set containing those objects which belong to both of them. We write it $X \cap Y$. In the inclusion criterion notation from above:

Definition 8 (Intersection)**.** $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$.

For example, $\{7, 8, 9\} \cap \{8, 9, 10\} = \{8, 9\}$.

It often helps to draw a picture. The two circles stand for X and Y, and their intersection is the shaded region where the circles overlap.

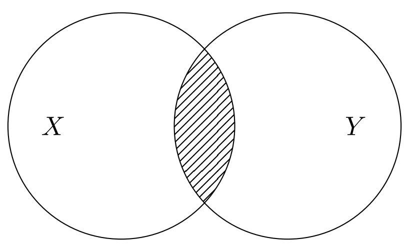

Figure 3: The intersection $X \cap Y$.

The first thing to notice is that taking an intersection only ever makes a set smaller, or leaves it as it was.

Proposition 8 (an intersection is a subset of each set)**.** *For any sets X and Y, $X \cap Y \subset X$ and $X \cap Y \subset Y$.*

Proof. Let x be an arbitrary element of $X \cap Y$. By the definition of intersection, $x \in X$ and $x \in Y$. Hence every element of $X \cap Y$ is an element of both X and Y. □

Intersection also respects the subset relation: making one of the two sets bigger cannot make the intersection smaller.

In Figure 4 the doubly shaded region is $X \cap Z$, and it sits inside $Y \cap Z$.

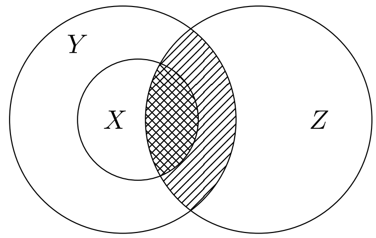

Figure 4: Proposition 9 (intersection respects inclusion): making a set bigger cannot make its intersection with Z smaller.

Proposition 9 (intersection respects inclusion)**.** *For any sets X, Y and Z, if $X \subset Y$, then $X \cap Z \subset Y \cap Z$.*

Proof. Suppose $x \in X \cap Z$. Then $x \in X$ and $x \in Z$. Since $X \subset Y$, we also have $x \in Y$. Hence $x \in Y \cap Z$. □

Intersection also behaves in some ways like the multiplication or addition of numbers. The order does not matter, and neither does the bracketing. (These properties are generally known as *commutativity* and *associativity* respectively.)

The first of these is our first chance to use the method of inclusion criteria.

Proposition 10 (intersection commutes)**.** *For any sets X and Y, $X \cap Y = Y \cap X$.*

Proof. The inclusion criterion for $X \cap Y$ is $x \in X$ and $x \in Y$. The inclusion criterion for $Y \cap X$ is $x \in Y$ and $x \in X$. These two criteria are logically equivalent, since a claim of the form "P and Q" holds in exactly the cases in which "Q and P" holds. That is, logical conjunction commutes. □

Notice that the fact we leaned on there is a fact about *logic*, and not a fact about sets. The reason $X \cap Y$ and $Y \cap X$ are the same set is that intersection is defined using "and", and conjunction commutes. The fact about sets is closely tied to the fact about logic, but it is the fact about logic which is prior.

We now have two general techniques for showing that two sets are identical, so let me set them side by side. The first method is to show that each set is a subset of the other, applying Proposition 1 (double inclusion). The second is to write down the inclusion criterion for each set and observe that the two criteria are equivalent.

From now on we will help ourselves to any fact about classical logic we need, without stopping to justify it. Proving such facts is the business of a logic course, not of this primer. You should feel free to do the same in your own proofs.

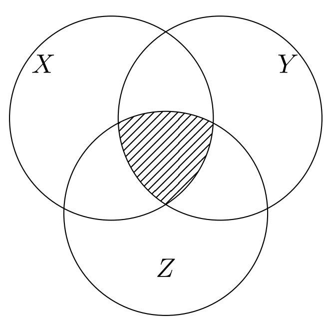

Figure 5: Proposition 11 (intersection is associative): the shaded region belongs to all three sets, however the brackets are placed.

Proposition 11 (intersection is associative)**.** *For any sets X, Y and Z, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.*

Proof. Write S_1 for the first set and S_2 for the second set. The inclusion criterion for S_1 is $x \in (X \cap Y)$ and $x \in Z$. The inclusion criterion for S_2 is $x \in X$ and $x \in (Y \cap Z)$. Applying the inclusion criteria for $X \cap Y$ and for $Y \cap Z$, the inclusion criterion for S_1 becomes ($x \in X$ and $x \in Y$) and $x \in Z$, while the inclusion criterion for S_2 becomes $x \in X$ and ($x \in Y$ and $x \in Z$). These two inclusion criteria are logically equivalent. □

Because of this last proposition we can simply write $X \cap Y \cap Z$, with no brackets and no ambiguity about what is meant. We can also ignore the difference between $X \cap Y$ and $Y \cap X$: the two are defined differently, but we have just proved that they must always contain exactly the same elements.

Two smaller facts are worth recording.

Proposition 12 (intersecting a set with itself)**.** *For any set X, $X \cap X = X$.*

Proof. The inclusion criterion for $X \cap X$ is $x \in X$ and $x \in X$, which is logically equivalent to $x \in X$ (the inclusion criterion for X). □

Proposition 13 (intersecting with the empty set)**.** *For any set X, $X \cap \varnothing = \varnothing$.*

Proof. If $x \in X \cap \varnothing$ then $x \in X$ and $x \in \varnothing$. But that is impossible, because $\varnothing$ contains no elements. Therefore there is no element of $X \cap \varnothing$, and so $X \cap \varnothing$ is equal to $\varnothing$ by Proposition 5 (the empty set is unique). □

Finally, intersection gives us another way of saying that one set is a subset of another.

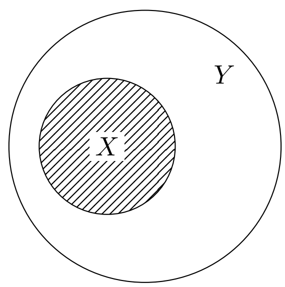

Figure 6: Proposition 14 (inclusion via intersection): when X lies inside Y, the shaded region $X \cap Y$ is all of X.

Proposition 14 (inclusion via intersection)**.** *For any sets X and Y, $X \subset Y$ if and only if $X \cap Y = X$.*

Proof. We show both directions of the biconditional.

(i) Left to right direction: suppose that $X \subset Y$. We want to show that $X \cap Y = X$. We have $X \cap Y \subset X$ by Proposition 8 (an intersection is a subset of each set). Now let x be an arbitrary element of X; since $X \subset Y$ we have $x \in Y$, and hence $x \in X \cap Y$. Since x was chosen arbitrarily, we can conclude $X \subset X \cap Y$. Putting these two subset claims together by Proposition 1 (double inclusion), we can conclude that $X \cap Y = X$.

(ii) Right to left direction: suppose that $X \cap Y = X$. We want to show that $X \subset Y$. Proposition 8 (an intersection is a subset of each set) implies that $X \cap Y \subset Y$, and since $X \cap Y = X$ we can conclude that $X \subset Y$, as required. □

That last proposition is our first *biconditional*: a claim of the form "P if and only if Q". To prove a biconditional we have to prove two things: that if P then Q, and that if Q then P. Each half is a conditional, so each half is proved in the way described above, by assuming the antecedent and deriving the consequent.

13 Unions

Given two sets X and Y, their *union* is the set containing those objects which belong to at least one of them. We write it $X \cup Y$. So the inclusion criterion for $X \cup Y$ is $x \in X$ or $x \in Y$. Where intersection asks for both, union asks for at least one. (Note that we use "or" in the inclusive sense: if x belongs to both X and Y, then we say it is true that "$x \in X$ or $x \in Y$".)

Definition 9 (Union)**.** $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$, where the "or" is inclusive.

For example, $\{7, 8, 9\} \cup \{8, 9, 10\} = \{7, 8, 9, 10\}$. In Figure 7, $X \cup Y$ is the shaded region: everything inside either circle, the overlap included.

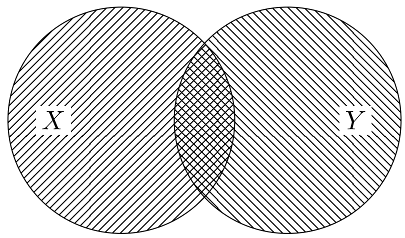

Figure 7: The union $X \cup Y$.

Where intersection only ever made a set smaller, union only ever makes it bigger.

Proposition 15 (each set is a subset of the union)**.** *For any sets X and Y, $X \subset X \cup Y$ and $Y \subset X \cup Y$.*

Proof. Suppose $x \in X$. Then we immediately have $x \in X$ or $x \in Y$; hence x satisfies the inclusion criterion for $X \cup Y$; hence $x \in X \cup Y$. Since x was chosen arbitrarily, it follows that $X \subset X \cup Y$.

The proof that $Y \subset X \cup Y$ is exactly analogous. □

The remaining facts about union are the mirror images of the ones we proved about intersection in Section 12, with "or" in place of "and". Rather than prove them for you, I will let you do them yourself. Each one can be done by the method of demonstrating equivalent inclusion criteria, and in each case the fact about logic you need is the one about "or" corresponding to the fact about "and" we used before.

Proposition 16 (union commutes)**.** *For any sets X and Y, $X \cup Y = Y \cup X$.*

Exercise 3. Prove Proposition 16 (union commutes).

Proposition 17 (union is associative)**.** *For any sets X, Y and Z, $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.*

Exercise 4. Prove Proposition 17 (union is associative).

From now on, we can write $X \cup Y \cup Z$ with no brackets, and we can ignore the difference between $X \cup Y$ and $Y \cup X$, exactly as we did for intersection. The next two propositions are the analogues of Propositions 12 (intersecting a set with itself) and 13 (intersecting with the empty set).

Proposition 18 (uniting a set with itself)**.** *For any set X, $X \cup X = X$.*

Exercise 5. Prove Proposition 18 (uniting a set with itself).

Proposition 19 (uniting with the empty set)**.** *For any set X, $X \cup \varnothing = X$.*

Exercise 6. Prove Proposition 19 (uniting with the empty set).

Proposition 20 (union respects inclusion)**.** *For any sets X, Y and Z, if $X \subset Y$, then $X \cup Z \subset Y \cup Z$.*

Exercise 7. Prove Proposition 20 (union respects inclusion).

Now for something which has no analogue among the propositions about intersection. Proposition 15 (each set is a subset of the union) tells us that $X \cup Y$ is big enough to contain both X and Y. The next proposition tells us that it is no bigger than it has to be: any set which contains both X and Y already contains their union.

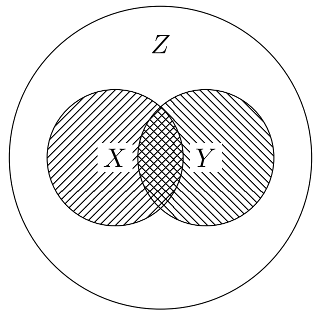

Figure 8: Proposition 21 (the union is the smallest set containing both): if Z contains both X and Y, it contains the whole shaded region.

Proposition 21 (the union is the smallest set containing both)**.** *For any sets X, Y and Z, if $X \subset Z$ and $Y \subset Z$, then $X \cup Y \subset Z$.*

Proof. Suppose that $X \subset Z$ and $Y \subset Z$. Let $x \in X \cup Y$. It follows that either (i) $x \in X$, or (ii) $x \in Y$.

Case (i): since $x \in X$ and $X \subset Z$, we have $x \in Z$.

Case (ii): since $x \in Y$ and $Y \subset Z$, we have $x \in Z$.

Hence $x \in Z$, from which we conclude that $X \cup Y \subset Z$. ☐

That proof needed a move we had not used before. To show that an arbitrary element of $X \cup Y$ lies in Z, we only know that it lies in X *or* in Y, and not which. So we take the two cases in turn and show that the conclusion follows in each of them. Since one case or the other must hold, the conclusion follows either way. This is called **proof by cases**, and it is what disjunctions generally demand of us. (If you have taken logic, you may recognise this as an informal version of the disjunction elimination inference rule.)

Finally, union gives us another way of saying that one set is a subset of another, just as intersection did in Proposition 14 (inclusion via intersection).

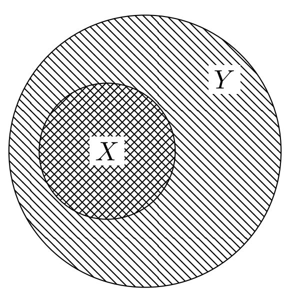

Figure 9: Proposition 22 (inclusion via union): when X lies inside Y, the shaded region $X \cup Y$ is all of Y.

Proposition 22 (inclusion via union)**.** *For any sets X and Y, $X \subset Y$ if and only if $X \cup Y = Y$.*

Proof. We again prove both sides of the biconditional.

(i) Left to right: suppose that $X \subset Y$. We want to show that $X \cup Y = Y$. We have $Y \subset X \cup Y$ immediately from Proposition 15 (each set is a subset of the union). Further, if $x \in X \cup Y$ then by the definition of union we have either (a) $x \in X$ or (b) $x \in Y$. In case (a), since $X \subset Y$ we have $x \in Y$. In case (b), of course we also have $x \in Y$. Hence $x \in Y$ either way, and so we conclude that $X \cup Y \subset Y$. By Proposition 1 (double inclusion) it follows that $X \cup Y = Y$.

(ii) Right to left: suppose that $X \cup Y = Y$. We want to show that $X \subset Y$. Proposition 15 (each set is a subset of the union) implies that $X \subset X \cup Y$, hence by substitution of identicals (that is, $X \cup Y = Y$) we have $X \subset Y$. □

14 Interactions between Unions and Intersections

So far we have taken union and intersection one at a time. When they are combined, each distributes over the other.

This is the first result which is genuinely hard to see just by reading the symbols, so here it is as a picture. On the left, we have shaded $Y \cup Z$ and then kept only the part inside X. On the right, we have shaded $X \cap Y$ and $X \cap Z$ separately. The two shaded regions are the same, and that is what the proposition says.

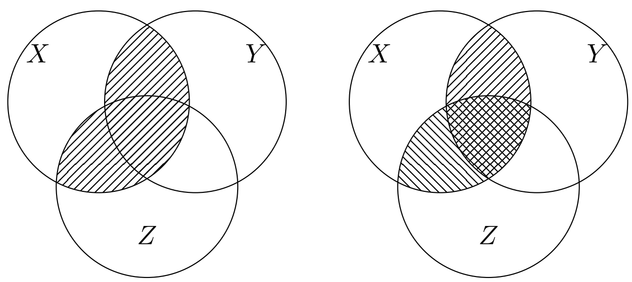

Figure 10: Proposition 23 (intersection distributes over union): $X \cap (Y \cup Z)$ on the left, $(X \cap Y) \cup (X \cap Z)$ on the right.

Proposition 23 (intersection distributes over union). *For any sets X, Y and Z, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.*

Proof. We will show that (i) $X \cap (Y \cup Z) \subset (X \cap Y) \cup (X \cap Z)$, and then that (ii) $(X \cap Y) \cup (X \cap Z) \subset X \cap (Y \cup Z)$. This is sufficient, by Proposition 1 (double inclusion).

(i): Suppose that $x \in X \cap (Y \cup Z)$. Then, by the definitions of intersection and of union, we have $x \in X$ AND either (a) $x \in Y$ or (b) $x \in Z$.

In case (a), we have $x \in X$ and $x \in Y$. Hence $x \in X \cap Y$, and then by Proposition 15 (each set is a subset of the union), $x \in (X \cap Y) \cup (X \cap Z)$.

In case (b), we have $x \in X$ and $x \in Z$. Hence $x \in X \cap Z$, and then by Proposition 15 (each set is a subset of the union), $x \in (X \cap Y) \cup (X \cap Z)$.

Either way, $x \in (X \cap Y) \cup (X \cap Z)$, so we can conclude that $X \cap (Y \cup Z) \subset (X \cap Y) \cup (X \cap Z)$.

(ii): Suppose that $x \in (X \cap Y) \cup (X \cap Z)$. By the definition of union, either (a) $x \in X \cap Y$ or (b) $x \in X \cap Z$.

In case (a), by the definition of intersection we have $x \in X$ and $x \in Y$. Since $x \in Y$, we have $x \in Y \cup Z$ by Proposition 15 (each set is a subset of the union). We thus have $x \in X$ and $x \in Y \cup Z$, and by the definition of intersection this implies that $x \in X \cap (Y \cup Z)$.

In case (b), by the definition of intersection we have $x \in X$ and $x \in Z$, and since Proposition 15 (each set is a subset of the union) also implies that $x \in Y \cup Z$, the rest of the proof is the same as in case (a).

Either way, we have $x \in X \cap (Y \cup Z)$, and it follows that $(X \cap Y) \cup (X \cap Z) \subset X \cap (Y \cup Z)$, as required.

$\square$

The next proposition is the same as the last, with union and intersection swapped over.

Figure 11 does for it what Figure 10 did for the last one.

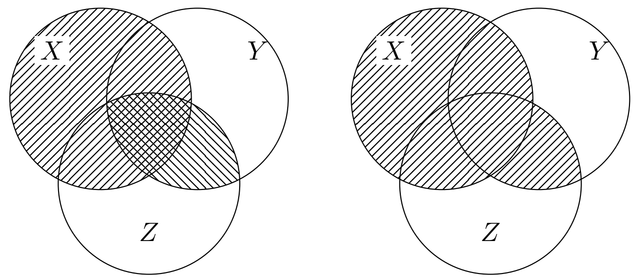

Figure 11: Proposition 24 (union distributes over intersection): $X \cup (Y \cap Z)$ on the left, $(X \cup Y) \cap (X \cup Z)$ on the right.

Proposition 24 (union distributes over intersection)**.** *For any sets X, Y and Z, $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.*

Exercise 8. Prove Proposition 24 (union distributes over intersection).

The next pair of identities are worth thinking about informally first. $X \cap Y$ is already part of X, so throwing it back in with X adds nothing new. And $X \cup Y$ contains the whole of X, so cutting it down to the part lying in X leaves X untouched. In Figure 12, the whole of X is shaded, and $X \cap Y$ is the doubly shaded part of it. Since that part is already inside X, putting it back in adds nothing.

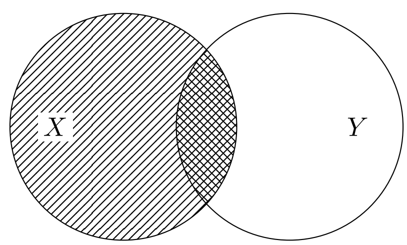

Figure 12: Propositions 25 (absorption for union) and 26 (absorption for intersection): all of X is shaded, and $X \cap Y$ is shaded twice.

Proposition 25 (absorption for union)**.** *For any sets X and Y, $X \cup (X \cap Y) = X$.*

Exercise 9. Prove Proposition 25 (absorption for union).

Hint: consider applying Proposition 8 (an intersection is a subset of each set) and Proposition 22 (inclusion via union).

Proposition 26 (absorption for intersection)**.** *For any sets X and Y, $X \cap (X \cup Y) = X$.*

Exercise 10. Prove Proposition 26 (absorption for intersection).

Hint: consider applying Proposition 14 (inclusion via intersection) and Proposition 15 (each set is a subset of the union).

15 Relative complements

Given two sets X and Y, the *complement of* Y *in* X is the set containing those objects which belong to X but not to Y. We write it $X \setminus Y$, and read it aloud as "X minus Y".[4] So the inclusion criterion for $X \setminus Y$ is $x \in X$ and $x \notin Y$, where '$x \notin Y$' abbreviates 'it is not the case that $x \in Y$':

Definition 10 (Relative complement)**.** $X \setminus Y = \{x : x \in X \text{ and } x \notin Y\}$, the *complement of* Y *in* X.

Notice that the order matters here, in a way that it did not for union and intersection.

For example, $\{7, 8, 9\} \setminus \{8, 9, 10\} = \{7\}$, whereas $\{8, 9, 10\} \setminus \{7, 8, 9\} = \{10\}$. In Figure 13, $X \setminus Y$ is the shaded region: the part of X lying outside Y.

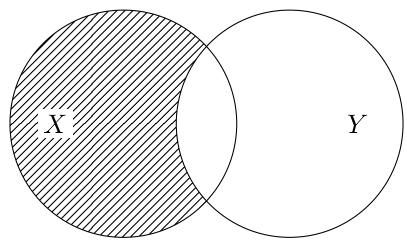

Figure 13: The relative complement $X \setminus Y$.

Taking a relative complement, like taking an intersection, only ever makes a set smaller.

Proposition 27 (a relative complement is a subset)**.** *For any sets X and Y, $X \setminus Y \subset X$.*

Proof. If $x \in X \setminus Y$ then $x \in X$ and $x \notin Y$. In particular, $x \in X$. □

Proposition 28 (subtracting the empty set)**.** *For any set X, $X \setminus \varnothing = X$.*

Exercise 11. Prove Proposition 28 (subtracting the empty set).

Proposition 29 (subtracting a set from itself)**.** *For any set X, $X \setminus X = \varnothing$.*

Exercise 12. Prove Proposition 29 (subtracting a set from itself).

16 Complements

In some contexts there is tacitly a domain: a largest set which everything we are discussing belongs to. If we are thinking about sets of real numbers, the domain might be $\mathbb{R}$; if we are doing probability theory, it might be the universal event containing all the possible outcomes.

When a domain U is understood, we write X^c as a shorthand for $U \setminus X$, and we call it simply the *complement* of X. So the inclusion criterion for X^c is $x \in U$ and $x \notin X$. Nothing here is new: X^c is the complement of X in U, with the domain left implicit:

Definition 11 (Complement)**.** Where a domain U is understood and $X \subset U$, the *complement* of X is $X^c = U \setminus X$.

In Figure 14, the rectangle is U and the shaded region is X^c.

[4]Many books instead call this the *difference* of X and Y, or the *set difference*, and some write it $X - Y$. It is all the same operation.

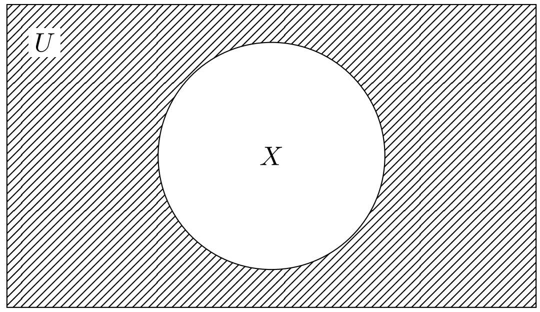

Figure 14: The complement X^c, relative to a domain U.

Throughout this section, X and Y are subsets of U.

Proposition 30 (a set and its complement exhaust the domain)**.** $X \cup X^c = U$.

Proof. We will apply Proposition 1 (double inclusion). For the first inclusion: we have already $X \subset U$, and it is clear from the definition that $X^c \subset U$. Hence, by Proposition 21 (the union is the smallest set containing both), $X \cup X^c \subset U$.

For the second inclusion: suppose that $x \in U$. By the law of excluded middle, either $x \in X$ or $x \notin X$. In the former case, $x \in X$, hence $x \in X \cup X^c$ by Proposition 15 (each set is a subset of the union). In the latter case, $x \in U$ and $x \notin X$, hence $x \in X^c$, and by Proposition 15 (each set is a subset of the union) $x \in X \cup X^c$. Therefore $U \subset X \cup X^c$. □

Proposition 31 (a set and its complement are disjoint)**.** $X \cap X^c = \varnothing$.

Proof. We will show by contradiction that there are no elements of $X \cap X^c$. Suppose there were some such element x. Then, by the definition of intersection and the inclusion criteria for X and X^c, we have $x \in X$ and $x \notin X$; a contradiction. □

Now try and prove the next three propositions yourself:

Proposition 32 (the complement of a complement)**.** $(X^c)^c = X$.

Exercise 13. Prove Proposition 32 (the complement of a complement).

Proposition 33 (the complement of the empty set)**.** $\varnothing^c = U$.

Exercise 14. Prove Proposition 33 (the complement of the empty set).

Proposition 34 (the complement of the domain)**.** $U^c = \varnothing$.

Exercise 15. Prove Proposition 34 (the complement of the domain).

Once a domain has been fixed, a relative complement can always be rewritten using the complement.

Proposition 35 (relative complement via complement)**.** $X \setminus Y = X \cap Y^c$.

Proof. The inclusion criterion for $X \setminus Y$ is $x \in X$ and $x \notin Y$. The inclusion criterion for $X \cap Y^c$ is $x \in X$ and $x \in U$ and $x \notin Y$. Since $X \subset U$, we have $x \in X$ if and only if $x \in X$ and $x \in U$ by Proposition 14 (inclusion via intersection). Hence the two inclusion criteria are equivalent. □

The next two results are named after the nineteenth-century mathematician Augustus De Morgan. They say that taking a complement turns intersection into union, and union into intersection. The shaded region in Figure 15 is everything outside $X \cap Y$. Reading it as one region gives $(X \cap Y)^c$; reading it as everything outside X together with everything outside Y gives $X^c \cup Y^c$.

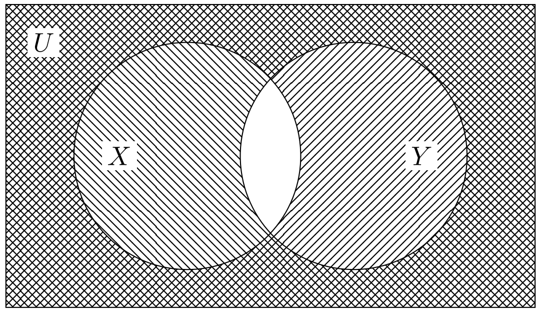

Figure 15: Proposition 36 (De Morgan, for intersection): the shaded region is both $(X \cap Y)^c$ and $X^c \cup Y^c$.

Proposition 36 (De Morgan, for intersection)**.** $(X \cap Y)^c = X^c \cup Y^c$.

Proof. We will prove this by showing that the inclusion criteria are logically equivalent.

The inclusion criterion for the set $(X \cap Y)^c$ is $x \in U$ and $x \notin X \cap Y$. Unpacking this further by the definition of intersection, the criterion is that (i) $x \in U$ and (ii) it is *not* the case that x is in both X and Y. Equivalently, x is in U *and* either (i) x is not in X, or (ii) x is not in Y.

The inclusion criterion for the set $X^c \cup Y^c$ is that x is in either X^c or Y^c. Thus, either (i) x is in U, but not in X, or (ii) x is in U, but not in Y. Equivalently, x is in U *and* either (i) x is not in X, or (ii) x is not in Y. But we have seen already that this is the same as the inclusion criterion for $(X \cap Y)^c$. $\square$

Notice that we tacitly helped ourselves to the corresponding *logical* De Morgan law here, stating that $\neg(X \wedge Y)$ is logically equivalent to $\neg X \vee \neg Y$. This can be broken down and proved from more basic logical steps; see any introductory logic textbook for the details. For our purposes here, since it's part of classical logic, we can simply make the move in one go without further explanation.

Now see if you can prove the dual De Morgan law yourself:

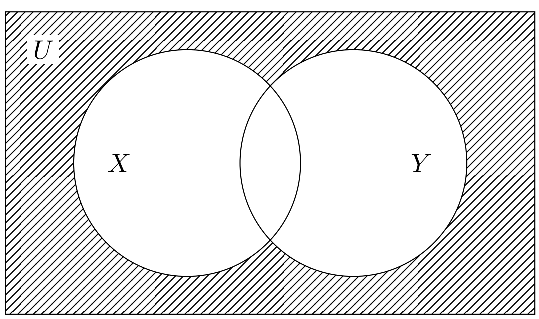

Figure 16: Proposition 37 (De Morgan, for union): the shaded region is both $(X \cup Y)^c$ and $X^c \cap Y^c$.

Proposition 37 (De Morgan, for union)**.** $(X \cup Y)^c = X^c \cap Y^c$.

Exercise 16. Prove Proposition 37 (De Morgan, for union).

One more useful fact is that complements reverse the direction of the subset relation.

In Figure 17, X^c is shaded and Y^c is shaded twice, so Y^c lies inside X^c even though X lies inside Y.

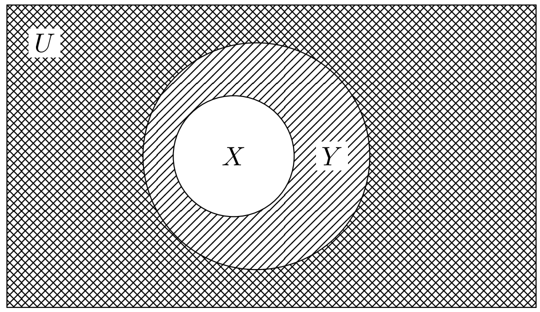

Figure 17: Proposition 38 (complements reverse inclusion): complements reverse inclusion.

Proposition 38 (complements reverse inclusion)**.** *$X \subset Y$ if and only if $Y^c \subset X^c$.*

Proof. Left to right: suppose that $X \subset Y$. If it were not the case that $Y^c \subset X^c$, then there would be some x which belongs to Y^c but not to X^c. In that case, we would have $x \in X$ but $x \notin Y$, contradicting that $X \subset Y$.

Right to left: this follows from the previous step of the argument if we apply Proposition 32 (the complement of a complement). (Do you see why?) □

17 Disjointness

Two sets are *disjoint* when they have no elements in common at all:

Definition 12 (Disjoint)**.** X and Y are *disjoint* when $X \cap Y = \varnothing$.

The paradigm case is a set and its complement. Proposition 31 (a set and its complement are disjoint) tells us that X and X^c are disjoint, and Proposition 30 (a set and its complement exhaust the domain) that between them they take up the whole of U.

Splitting a set into disjoint pieces can be useful in certain contexts, so it's worth getting used to doing it. The reason is that pieces which do not overlap can be counted or measured separately without anything being counted twice.

Taking a relative complement is the basic way of producing a pair of disjoint sets.

Proposition 39 (a set is disjoint from what is taken off another)**.** *For any sets X and Y, X and $Y \setminus X$ are disjoint.*

Proof. Suppose for contradiction that there is some $x \in X \cap (Y \setminus X)$. Then we have $x \in X$ by the definition of intersection and $x \notin X$ by the definition of intersection and relative complement. These two statements are contradictory. □

Disjointness is also inherited by subsets: if two sets do not overlap, then neither do any smaller sets taken from them.

Proposition 40 (disjointness passes to subsets)**.** *For any sets X, Y and Z, if X and Y are disjoint and $Z \subset X$, then Z and Y are disjoint.*

Proof. Suppose that X and Y are disjoint and that $Z \subset X$. Since $Z \subset X$, Proposition 9 (intersection respects inclusion) gives us $Z \cap Y \subset X \cap Y$. But $X \cap Y = \varnothing$, since X and Y are disjoint, so $Z \cap Y \subset \varnothing$. By Proposition 6 (the only subset of $\varnothing$), then, $Z \cap Y = \varnothing$; that is, Z and Y are disjoint. □

The next three propositions give us ways of decomposing a set into two disjoint components. In the first two, the components are disjoint by Proposition 39 (a set is disjoint from what is taken off another). In the third they are disjoint by Propositions 39 (a set is disjoint from what is taken off another) and 40 (disjointness passes to subsets) together, since $X \cap Y \subset X$.

Each of the propositions has a diagram to help you understand. In the first, X sits inside Y, and the two pieces are X itself and the ring around it.

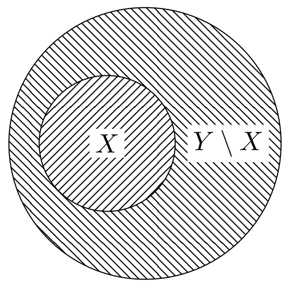

Figure 18: Proposition 41 (splitting a set around a subset): Y split into X and $Y \setminus X$.

Proposition 41 (splitting a set around a subset)**.** *For any sets X and Y, if $X \subset Y$, then $Y = X \cup (Y \setminus X)$.*

Proof. Suppose that $X \subset Y$. We will show that $Y = X \cup (Y \setminus X)$ using Proposition 1 (double inclusion).

First, let us show that $Y \subset X \cup (Y \setminus X)$. Let $y \in Y$. By the law of excluded middle, either (i) $y \in X$ or (ii) $y \notin X$. Case (i): if $y \in X$, then $y \in X \cup (Y \setminus X)$ by the definition of union. Case (ii): if $y \notin X$, then $y \in Y \setminus X$ by the definition of relative complement; hence $y \in X \cup (Y \setminus X)$ by the definition of union.

Next we will show that $X \cup (Y \setminus X) \subset Y$. Suppose that $x \in X \cup (Y \setminus X)$. There are two cases: either (i) $x \in X$ or (ii) $x \in Y \setminus X$. Case (i): if $x \in X$, then since $X \subset Y$ we have $x \in Y$. Case (ii): if $x \in Y \setminus X$, then by definition of relative complement, $x \in Y$. $\square$

In the second, we take all of X and only the part of Y which is new.

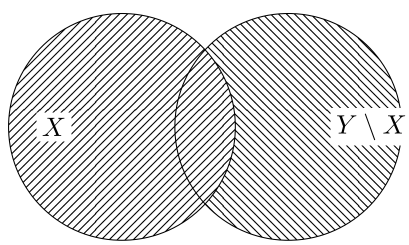

Figure 19: Proposition 42 (splitting a union into disjoint pieces): $X \cup Y$ split into X and $Y \setminus X$.

Proposition 42 (splitting a union into disjoint pieces)**.** *For any sets X and Y, $X \cup Y = X \cup (Y \setminus X)$.*

Exercise 17. Prove Proposition 42 (splitting a union into disjoint pieces).

In the third, Y alone is cut into the part outside X and the part inside it.

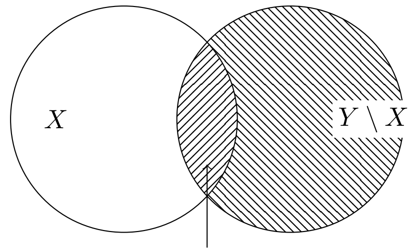

Figure 20: Proposition 43 (splitting a set by another set): Y split into $Y \setminus X$ and $X \cap Y$.

Proposition 43 (splitting a set by another set)**.** *For any sets X and Y, $Y = (Y \setminus X) \cup (X \cap Y)$.*

Exercise 18. Prove Proposition 43 (splitting a set by another set).

One last fact about disjointness, which we will want later.

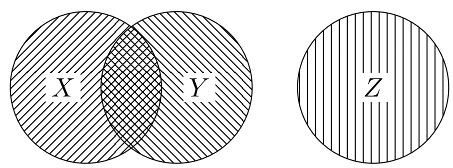

Figure 21: Proposition 44 (a union of sets disjoint from Z is disjoint from Z): if neither X nor Y meets Z, their union does not meet it either.

Proposition 44 (a union of sets disjoint from Z is disjoint from Z)**.** *For any sets X, Y and Z, if X and Z are disjoint and Y and Z are disjoint, then $X \cup Y$ and Z are disjoint.*

Proof. By Proposition 23 (intersection distributes over union), $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$. As X, Z and Y, Z are both disjoint pairs, $X \cap Z = Y \cap Z = \varnothing$, hence $(X \cup Y) \cap Z = \varnothing \cup \varnothing$. By Proposition 19 (uniting with the empty set), $\varnothing \cup \varnothing = \varnothing$, therefore $(X \cup Y)$ and Z are disjoint. $\square$

18 Collections of sets

We saw in Section 4 that a set can have other sets among its elements. This section collects a few things worth saying about sets of that kind.

Given a set X, all of the subsets of X can themselves be gathered into a single set. We call it the *power set* of X, and we write it $\mathcal{P}(X)$. So the inclusion criterion for $\mathcal{P}(X)$ is $Y \subset X$:

Definition 13 (Power set)**.** $\mathcal{P}(X) = \{Y : Y \subset X\}$.

For example, $\mathcal{P}(\{7, 8\}) = \{\varnothing, \{7\}, \{8\}, \{7, 8\}\}$. Notice that $\varnothing$ and X are elements of $\mathcal{P}(X)$ whatever X is, by Propositions 4 (an empty set is a subset of anything) and 2 (every set is a subset of itself).

We often want to take the union or the intersection of more than two sets at once. Let $\mathcal{F}$ be a collection of sets, by which I just mean a set whose elements are themselves sets:

Definition 14 (Collection)**.** A *collection* is a set whose elements are themselves sets.

The *union* of $\mathcal{F}$ is the set of objects belonging to at least one element of $\mathcal{F}$, and we write it $\bigcup \mathcal{F}$. The *intersection* of $\mathcal{F}$ is the set of objects belonging to every element of $\mathcal{F}$, and we write it $\bigcap \mathcal{F}$. So the inclusion criterion for $\bigcup \mathcal{F}$ is that $x \in X$ for at least one $X \in \mathcal{F}$, and the inclusion criterion for $\bigcap \mathcal{F}$ is that $x \in X$ for every $X \in \mathcal{F}$:

Definition 15 (Union of a collection)**.** $\bigcup \mathcal{F} = \{x : x \in X \text{ for at least one } X \in \mathcal{F}\}$.

Definition 16 (Intersection of a collection)**.** $\bigcap \mathcal{F} = \{x : x \in X \text{ for every } X \in \mathcal{F}\}$.

For example, $\bigcup\{\{7,8\},\{8,9\},\{9,10\}\} = \{7,8,9,10\}$. On the other hand, $\bigcap\{\{7,8\},\{8,9\},\{9,10\}\} = \varnothing$, since no number is in all three. And $\bigcap\{\{7,8,9\},\{8,9\},\{9,10\}\} = \{9\}$.

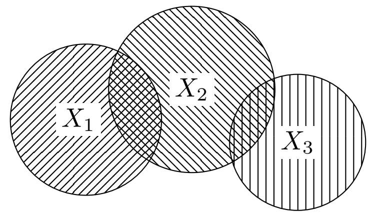

Figure 22: The union of a collection $\mathcal{F} = \{X_1, X_2, X_3\}$ is the whole shaded region.

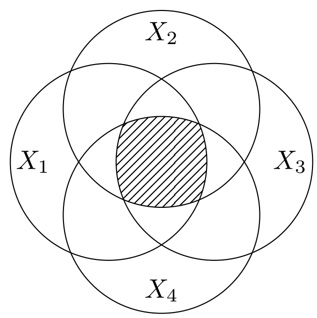

Figure 23: The intersection of a collection $\mathcal{F} = \{X_1, X_2, X_3, X_4\}$ is the shaded region.

Taking $\mathcal{F}$ to be $\{X, Y\}$ gives us back the operations we already have: $\bigcup\{X,Y\} = X \cup Y$ and $\bigcap\{X,Y\} = X \cap Y$. We will always assume that $\mathcal{F}$ is non-empty when taking an intersection.

Proposition 45 (each set is a subset of the union of its collection)**.** *For any collection $\mathcal{F}$ and any $X \in \mathcal{F}$, $X \subset \bigcup \mathcal{F}$.*

Proof. Suppose that $x \in X$. By the definition of union, for any $Y \in \mathcal{F}$, if $y \in Y$ then $y \in \bigcup \mathcal{F}$. In particular, since $X \in \mathcal{F}$, we have $x \in \bigcup \mathcal{F}$. □

Proposition 46 (the intersection of a collection is a subset of each)**.** *For any collection $\mathcal{F}$ and any $X \in \mathcal{F}$, $\bigcap \mathcal{F} \subset X$.*

Proof. Suppose that $x \in \bigcap \mathcal{F}$. Then, by the definition of intersection, for each $Y \in \mathcal{F}$, we have $x \in Y$. In particular, since $X \in \mathcal{F}$, we have $x \in X$. □

These are the general versions of Propositions 15 (each set is a subset of the union) and 8 (an intersection is a subset of each set).

In practice you will often see these written with an index. If we have a set X_i for each i in some set I, people write $\bigcup_{i \in I} X_i$ for $\bigcup\{X_i : i \in I\}$, and $\bigcap_{i \in I} X_i$ for $\bigcap\{X_i : i \in I\}$. This is just a way of labelling the elements of the collection, and nothing new is going on.

19 Partitions

A *partition* of a set X is a collection $\mathcal{F}$ of subsets of X satisfying two conditions:

Definition 17 (Partition)**.** A *partition* of X is a collection $\mathcal{F}$ of subsets of X such that (i) any two distinct elements of $\mathcal{F}$ are disjoint, and (ii) $\bigcup \mathcal{F} = X$.

The idea is that a partition cuts X into pieces, with nothing left out and no overlaps.

For example, $\{\{7,8\},\{9\},\{10,11\}\}$ is a partition of $\{7,8,9,10,11\}$. But $\{\{7,8\},\{8,9\}\}$ is not a partition of $\{7,8,9\}$, because the two pieces overlap; and $\{\{7,8\},\{9\}\}$ is not a partition of $\{7,8,9,10\}$, because 10 is left out.

A picture of a partition of X into four pieces:

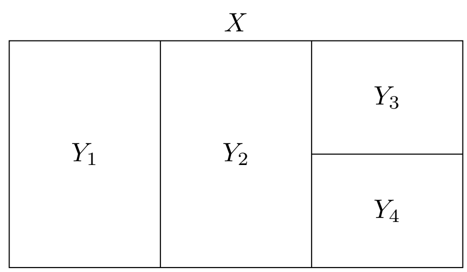

Figure 24: A partition of X into four pieces.

Proposition 47 (each element lies in exactly one cell)**.** *If $\mathcal{F}$ is a partition of X, then every element of X belongs to exactly one element of $\mathcal{F}$.*

Proof. Let x be an arbitrary element of X. We want to show that x belongs to exactly one element of $\mathcal{F}$.

First, we show that x belongs to at least one element of $\mathcal{F}$. By part (ii) of the definition of a partition, we have $x \in \bigcup \mathcal{F}$, hence by definition of union there is some $Y \in \mathcal{F}$ such that $x \in Y$.

Next, we show that x belongs to at most one element of $\mathcal{F}$. Suppose for contradiction that x belongs to two distinct elements of $\mathcal{F}$, which we can denote by Y and Z. Then, by the definition of pairwise intersection, we have $x \in Y \cap Z$. Hence $Y \cap Z \neq \varnothing$, contradicting condition (i), which implies that Y and Z are disjoint. $\square$

The decompositions in Section 17 were partitions into two pieces.

20 Summary of results

In Propositions 30 to 38, X and Y are subsets of a domain U.

Proposition 1 (double inclusion). For any sets X and Y, if $X \subset Y$ and $Y \subset X$, then $X = Y$.

Proposition 2 (every set is a subset of itself). Every set is a subset of itself.

Proposition 3 (transitivity of inclusion). For any sets X, Y and Z, if $X \subset Y$ and $Y \subset Z$, then $X \subset Z$.

Proposition 4 (an empty set is a subset of anything). For any sets X and Y, if X contains no elements, then $X \subset Y$.

Proposition 5 (the empty set is unique). Any two sets with no elements are identical.

Proposition 6 (the only subset of $\varnothing$). For any set X, if $X \subset \varnothing$, then $X = \varnothing$.

Proposition 7 (equivalent criteria give the same set). If two sets have equivalent inclusion criteria, then they are identical.

Proposition 8 (an intersection is a subset of each set). For any sets X and Y, $X \cap Y \subset X$ and $X \cap Y \subset Y$.

Proposition 9 (intersection respects inclusion). For any sets X, Y and Z, if $X \subset Y$, then $X \cap Z \subset Y \cap Z$.

Proposition 10 (intersection commutes). For any sets X and Y, $X \cap Y = Y \cap X$.

Proposition 11 (intersection is associative). For any sets X, Y and Z, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

Proposition 12 (intersecting a set with itself). For any set X, $X \cap X = X$.

Proposition 13 (intersecting with the empty set). For any set X, $X \cap \varnothing = \varnothing$.

Proposition 14 (inclusion via intersection). For any sets X and Y, $X \subset Y$ if and only if $X \cap Y = X$.

Proposition 15 (each set is a subset of the union). For any sets X and Y, $X \subset X \cup Y$ and $Y \subset X \cup Y$.

Proposition 16 (union commutes). For any sets X and Y, $X \cup Y = Y \cup X$.

Proposition 17 (union is associative). For any sets X, Y and Z, $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.

Proposition 18 (uniting a set with itself). For any set X, $X \cup X = X$.

Proposition 19 (uniting with the empty set). For any set X, $X \cup \varnothing = X$.

Proposition 20 (union respects inclusion). For any sets X, Y and Z, if $X \subset Y$, then $X \cup Z \subset Y \cup Z$.

Proposition 21 (the union is the smallest set containing both). For any sets X, Y and Z, if $X \subset Z$ and $Y \subset Z$, then $X \cup Y \subset Z$.

Proposition 22 (inclusion via union). For any sets X and Y, $X \subset Y$ if and only if $X \cup Y = Y$.

Proposition 23 (intersection distributes over union). For any sets X, Y and Z, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

Proposition 24 (union distributes over intersection). For any sets X, Y and Z, $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Proposition 25 (absorption for union). For any sets X and Y, $X \cup (X \cap Y) = X$.

Proposition 26 (absorption for intersection). For any sets X and Y, $X \cap (X \cup Y) = X$.

Proposition 27 (a relative complement is a subset). For any sets X and Y, $X \setminus Y \subset X$.

Proposition 28 (subtracting the empty set). For any set X, $X \setminus \varnothing = X$.

Proposition 29 (subtracting a set from itself). For any set X, $X \setminus X = \varnothing$.

Proposition 30 (a set and its complement exhaust the domain). $X \cup X^c = U$.

Proposition 31 (a set and its complement are disjoint). $X \cap X^c = \varnothing$.

Proposition 32 (the complement of a complement). $(X^c)^c = X$.

Proposition 33 (the complement of the empty set). $\varnothing^c = U$.

Proposition 34 (the complement of the domain). $U^c = \varnothing$.

Proposition 35 (relative complement via complement). $X \setminus Y = X \cap Y^c$.

Proposition 36 (De Morgan, for intersection). $(X \cap Y)^c = X^c \cup Y^c$.

Proposition 37 (De Morgan, for union). $(X \cup Y)^c = X^c \cap Y^c$.

Proposition 38 (complements reverse inclusion). $X \subset Y$ if and only if $Y^c \subset X^c$.

Proposition 39 (a set is disjoint from what is taken off another). For any sets X and Y, X and $Y \setminus X$ are disjoint.

Proposition 40 (disjointness passes to subsets). For any sets X, Y and Z, if X and Y are disjoint and $Z \subset X$, then Z and Y are disjoint.

Proposition 41 (splitting a set around a subset). For any sets X and Y, if $X \subset Y$, then $Y = X \cup (Y \setminus X)$.

Proposition 42 (splitting a union into disjoint pieces). For any sets X and Y, $X \cup Y = X \cup (Y \setminus X)$.

Proposition 43 (splitting a set by another set). For any sets X and Y, $Y = (Y \setminus X) \cup (X \cap Y)$.

Proposition 44 (a union of sets disjoint from Z is disjoint from Z). For any sets X, Y and Z, if X and Z are disjoint and Y and Z are disjoint, then $X \cup Y$ and Z are disjoint.

Proposition 45 (each set is a subset of the union of its collection). For any collection $\mathcal{F}$ and any $X \in \mathcal{F}$, $X \subset \bigcup \mathcal{F}$.

Proposition 46 (the intersection of a collection is a subset of each). For any collection $\mathcal{F}$ and any $X \in \mathcal{F}$, $\bigcap \mathcal{F} \subset X$.

Proposition 47 (each element lies in exactly one cell). If $\mathcal{F}$ is a partition of X, then every element of X belongs to exactly one element of $\mathcal{F}$.

21 List of Exercises

In each exercise you may assume every proposition stated up to that point in the primer, and use it in your proof. You may not assume any proposition which comes later.

In Exercises 13 to 16, X and Y are subsets of a domain U.

Exercise 1. Prove Proposition 2 (every set is a subset of itself):

Every set is a subset of itself.

Exercise 2. Prove Proposition 5 (the empty set is unique), as a corollary of Propositions 1 (double inclusion) and 4 (an empty set is a subset of anything):

Any two sets with no elements are identical.

Exercise 3. Prove Proposition 16 (union commutes):

For any sets X and Y, $X \cup Y = Y \cup X$.

Exercise 4. Prove Proposition 17 (union is associative):

For any sets X, Y and Z, $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.

Exercise 5. Prove Proposition 18 (uniting a set with itself):

For any set X, $X \cup X = X$.

Exercise 6. Prove Proposition 19 (uniting with the empty set):

For any set X, $X \cup \varnothing = X$.

Exercise 7. Prove Proposition 20 (union respects inclusion):

For any sets X, Y and Z, if $X \subset Y$, then $X \cup Z \subset Y \cup Z$.

Exercise 8. Prove Proposition 24 (union distributes over intersection):

For any sets X, Y and Z, $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

Exercise 9. Prove Proposition 25 (absorption for union):

For any sets X and Y, $X \cup (X \cap Y) = X$.

Hint: consider applying Proposition 8 (an intersection is a subset of each set) and Proposition 22 (inclusion via union).

Exercise 10. Prove Proposition 26 (absorption for intersection):

For any sets X and Y, $X \cap (X \cup Y) = X$.

Hint: consider applying Proposition 14 (inclusion via intersection) and Proposition 15 (each set is a subset of the union).

Exercise 11. Prove Proposition 28 (subtracting the empty set):

For any set X, $X \setminus \varnothing = X$.

Exercise 12. Prove Proposition 29 (subtracting a set from itself):

For any set X, $X \setminus X = \varnothing$.

Exercise 13. Prove Proposition 32 (the complement of a complement):

$(X^c)^c = X$.

Exercise 14. Prove Proposition 33 (the complement of the empty set):

$\varnothing^c = U$.

Exercise 15. Prove Proposition 34 (the complement of the domain):

$U^c = \varnothing$.

Exercise 16. Prove Proposition 37 (De Morgan, for union):

$(X \cup Y)^c = X^c \cap Y^c$.

Exercise 17. Prove Proposition 42 (splitting a union into disjoint pieces):

For any sets X and Y, $X \cup Y = X \cup (Y \setminus X)$.

Exercise 18. Prove Proposition 43 (splitting a set by another set):

For any sets X and Y, $Y = (Y \setminus X) \cup (X \cap Y)$.