PE3101P: Decision Theory and Social Choice

Tutorial Exercises 1

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Instructions

These questions cover weeks 1 to 3. They are not assessed.

Throughout, you should assume that credences obey the probability axioms, unless the possibility that they do not is stated or raised.

Question 1

Recall that, for any two acts A and B over the same set of states of nature:

– A **strongly statewise dominates** B if A results in a better outcome than B in *every* state of nature.

– A **weakly statewise dominates** B if A results in an outcome at least as good as B's in *every* state of nature, and a better outcome in *at least one* state of nature.

In parts (a) to (c), give an example of the kind of decision table asked for. A small table of utilities is enough; you do not need to describe a situation. In each case, say briefly why your example has the feature asked for.

(a) A decision table with two acts and two states of nature, in which one act weakly but not strongly statewise dominates the other.

(b) A decision table with three acts, in which no act statewise dominates any other, either weakly or strongly — and in which no two acts have the same utility in every state.

(c) A decision table with three acts A, B and C, in which A weakly but not strongly statewise dominates B, and B strongly statewise dominates C.

(d) State whether, in your answer to part (c), A strongly statewise dominates C, A weakly but not strongly statewise dominates C, or A does not statewise dominate C.

(e) Could your answer to part (d) have come out differently? If so, provide a decision table exhibiting this different answer: that is, a decision problem satisfying the conditions in part (c), but for which the answer to part (d) is different. If not, explain why.

Question 2

Throughout this question, the sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\},$$

the six possible results of rolling a die.

Recall that a collection of events is **mutually exclusive** when no two of the events can obtain together — that is, when the events are pairwise disjoint — and **jointly exhaustive** when every outcome in Ω belongs to at least one of the events.

For each of the following collections of events, say whether they are

- **mutually exclusive** only;
- **jointly exhaustive** only;
- **both**; or
- **neither**.

(a) $A = \{1, 2, 3, 4\}, \quad B = \{5, 6\}.$

(b) $A = \{1, 6\}, \quad B = \{2, 5\}.$

(c) $A = \{1, 2, 3, 4\}, \quad B = \{4, 5, 6\}.$

(d) $A = \{2, 3, 4, 5\}, \quad B = \{4, 5, 6\}.$

(e) $A = \{1, 6\}, \quad B = \{2, 5\}, \quad C = \{3, 4\}.$

(f) $A = \{1, 2\}, \quad B = \{3, 4, 5\}, \quad C = \{5, 6\}.$

(g) $A = \{2, 4, 6\}, \quad B = \{1, 5\}.$

(h) $A = \{1, 2\}, \quad B = \{2, 3, 4\}, \quad C = \{4, 6\}.$

Question 3

In each of the following, exactly one of the four claims is **guaranteed** to hold: it is true for every credence function satisfying the probability axioms, and for every pair of events A and B with positive credence. Each of the other three claims fails for some credence function. Say which claim is the guaranteed one.

(a) (i) $\mathrm{Cr}(A \mid B) + \mathrm{Cr}(A \mid \neg B) = 1$

(ii) $\mathrm{Cr}(A \mid B) \geq \mathrm{Cr}(A)$

(iii) $\mathrm{Cr}(A \mid B) + \mathrm{Cr}(\neg A \mid B) = 1$

(iv) $\mathrm{Cr}(A \mid B) = \mathrm{Cr}(B \mid A)$

(b) (i) $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(B \mid A) \cdot \mathrm{Cr}(A)$

(ii) $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(A) \cdot \mathrm{Cr}(B)$

(iii) $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(B \mid A) \cdot \mathrm{Cr}(B)$

(iv) $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(A) + \mathrm{Cr}(B) - 1$

(c) (i) $\mathrm{Cr}(A \cup B) = 1 - \mathrm{Cr}(A \cap B)$

(ii) $\mathrm{Cr}(A \cup B) \leq \mathrm{Cr}(A) + \mathrm{Cr}(B)$

(iii) $\mathrm{Cr}(A \cup B) = \mathrm{Cr}(A) + \mathrm{Cr}(B)$

(iv) $\mathrm{Cr}(A \cup B) \geq \mathrm{Cr}(A) + \mathrm{Cr}(B)$

Question 4

Recall the probability axioms. For any events A and B:

– **Non-negativity:** $\mathrm{Cr}(A) \geq 0$.

– **Normalisation:** $\mathrm{Cr}(\Omega) = 1$, where Ω is the sample space.

– **Additivity:** if A and B are disjoint, then $\mathrm{Cr}(A \cup B) = \mathrm{Cr}(A) + \mathrm{Cr}(B)$.

In each of the following, an agent's credences in some events are given, together with facts about the events. Say whether the credences can all be held together without violating the axioms. If they cannot, say which axiom or axioms are violated, and explain why. If one of two or more axioms must be violated, but it is not settled which one, indicate this clearly.

Recall that, for any events X and Y, the event Y is the union of the disjoint events $X \cap Y$ and $Y \setminus X$ — where $Y \setminus X$ is the event containing the outcomes that are in Y but not in X.

(a) $\mathrm{Cr}(P) = 0.4$, $\mathrm{Cr}(\neg P) = 0.7$.

(b) P and Q are mutually exclusive; $\mathrm{Cr}(P) = 0.3$, $\mathrm{Cr}(Q) = 0.45$, $\mathrm{Cr}(P \cup Q) = 0.75$.

(c) $P \subset Q$, so that every outcome in P is also in Q; $\mathrm{Cr}(P) = 0.5$, $\mathrm{Cr}(Q) = 0.3$.

(d) $\mathrm{Cr}(P) = 0.5$, $\mathrm{Cr}(Q) = 0.5$, $\mathrm{Cr}(P \cap Q) = 0.2$, $\mathrm{Cr}(P \cup Q) = 0.9$.

Question 5

In each of the following, somebody reasons to a conclusion about the probability of something. Say whether the reasoning is correct. If it is not, say what has gone wrong.

(a) "A fair die is to be rolled. The probability that the roll is even is $1/2$, and the probability that it is at least 4 is $1/2$. These are two different ways the roll could be, so by additivity the probability that it is even, or at least 4, must be $1/2 + 1/2 = 1$."

(b) "The probability that the home team wins tomorrow's match is 0.5. A team cannot win a match without scoring at least once. So the probability that the home team scores at least once is at least 0.5."

(c) "The probability that the 8 am bus is full is 0.2, and the probability that the 9 am bus is full is 0.2. So the probability that both are full is $0.2 \times 0.2 = 0.04$."

(d) "The probability that the train is delayed, given that it snows, is 0.8; and the probability that the train is delayed, given that it does not snow, is 0.15. Snow and no snow are the only two possibilities, so the probability that the train is delayed is $0.8 + 0.15 = 0.95$."

Question 6

Recall that events A and B are *mutually exclusive* when $A \cap B = \varnothing$, and that they are *probabilistically independent* when $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(A) \cdot \mathrm{Cr}(B)$.

You may assume that $\mathrm{Cr}(\varnothing) = 0$, and that if $X \subset Y$ then $\mathrm{Cr}(X) \leq \mathrm{Cr}(Y)$.

(a) Show that if A and B are mutually exclusive, and $\mathrm{Cr}(A) > 0$ and $\mathrm{Cr}(B) > 0$, then A and B are **not** probabilistically independent.

(b) Show that if $\mathrm{Cr}(A) = 0$, then A and B are probabilistically independent.

Question 7

Recall two ways of saying that events A and B are probabilistically independent. The first way is:

$$\mathrm{Cr}(A \cap B) = \mathrm{Cr}(A) \cdot \mathrm{Cr}(B).$$

The second says that learning B makes no difference to how likely A is:

$$\mathrm{Cr}(A \mid B) = \mathrm{Cr}(A).$$

Recall also the definition of conditional probability:

$$\mathrm{Cr}(A \mid B) = \frac{\mathrm{Cr}(A \cap B)}{\mathrm{Cr}(B)}.$$

Assume throughout that $\mathrm{Cr}(B) > 0$.

(a) Show that if $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(A) \cdot \mathrm{Cr}(B)$, then $\mathrm{Cr}(A \mid B) = \mathrm{Cr}(A)$.

(b) Show the converse: if $\mathrm{Cr}(A \mid B) = \mathrm{Cr}(A)$, then $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(A) \cdot \mathrm{Cr}(B)$.

Question 8

A fair six-sided die is to be rolled, so that the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and you assign each outcome credence $1/6$. Consider the following four events.

$$A = \{1, 3, 5\}, \qquad B = \{3, 4, 5, 6\}, \qquad C = \{5, 6\}, \qquad D = \{2\}.$$

Identify **every** pair of these events that is probabilistically independent, and show your working in each case — including for the pairs that are not.

Question 9

Recall the betting interpretation of credences. Write an *\$S bet on A* for a deal that pays $\$S$ if A is true and nothing otherwise.

The betting interpretation of credences. An agent whose credence in A is p will buy an $\$S$ bet on A for any price below $\$pS$, and will sell one for any price above $\$pS$.

Buying a bet means paying money in order to receive the uncertain payout; selling a bet means receiving money in return for taking on the risk of the uncertain loss. Prices are paid when a bet is agreed; the payout is settled afterwards, once it is known whether the event obtained.

Assume the betting interpretation throughout. In each of the following, the agent's credences violate one of the probability axioms. In each case:

- say which axiom her credences violate;
- state a single bet, **with its price**, that the agent will accept according to the betting interpretation; and
- show that, having accepted it, she loses money *whatever happens*.

Take the stake to be $S = 100$ throughout. A negative price means that the money flows the other way: an agent who *sells* a bet at a price of $-\$20$ pays \$20 to the buyer, and still owes the $\$S$ payout if the event obtains.

(a) $\mathrm{Cr}(A) = -0.4$, for some event A.

(b) $\mathrm{Cr}(\Omega) = 0.9$, where Ω is the sample space.

(c) $\mathrm{Cr}(\Omega) = 1.2$.

Question 10

Recall that, in a decision problem, a **dependency hypothesis** lists, for each act available, which outcome would result were that act performed.

It will help to have a compact notation: write $\langle o, o' \rangle$ for the dependency hypothesis on which the first act listed would result in outcome o, and the second act in outcome o'.

Each of the following describes a decision situation in words. The outcomes have been given letters, for your convenience in writing the dependency hypotheses. In each case, first identify the acts available and the possible outcomes; then list **every** dependency hypothesis.

(a) You are deciding whether to take the bus to campus or to walk. Whichever you do, either you will arrive on time (a), or you will be late (b).

(b) You have a headache, and you are deciding whether to take a painkiller. Either the headache will be (g)one within the hour, or it will (n)ot be gone.

(c) You have one seedling, and you are deciding whether to plant it in the garden bed or to keep it in a pot on the balcony. Either it will (f)lower this year, or it will (n)ot.

Question 11

An agent faces a choice between two acts, A_1 and A_2. There are two states of nature, S_1 and S_2, and **these states are causally independent of the acts**. Her utilities are as follows.

	S_1	S_2
A_1	20	0
A_2	22	2

Her credences are

$$\mathrm{Cr}(S_1 \mid A_1) = 0.8, \qquad \mathrm{Cr}(S_1 \mid A_2) = 0.2,$$
$$\mathrm{Cr}(A_1) = 0.75, \qquad \mathrm{Cr}(A_2) = 0.25.$$

She is certain that she will perform exactly one of the two acts.

(a) Using the law of total probability, find $\mathrm{Cr}(S_1)$ and $\mathrm{Cr}(S_2)$.

(b) Find $\mathrm{Cr}(S_2 \mid A_1)$ and $\mathrm{Cr}(S_2 \mid A_2)$. (Note that, as S_1 and S_2 are states of nature, they are mutually exclusive and jointly exhaustive.)

(c) Taking the states of nature to be S_1 and S_2 as given above, does either act statewise dominate the other? If so, which, and is the dominance strong or weak?

(d) The *causal* expected utility of an act A, where the states are causally independent of the acts, is

$$\mathrm{EU}_c(A) = \sum_i \mathrm{Cr}(S_i)\, U(A, S_i).$$

Calculate $\mathrm{EU}_c(A_1)$ and $\mathrm{EU}_c(A_2)$. Which act does the causal theory recommend?

(e) The *evidential* expected utility of an act A is

$$\mathrm{EU}_e(A) = \sum_i \mathrm{Cr}(S_i \mid A)\, U(A, S_i).$$

Calculate $\mathrm{EU}_e(A_1)$ and $\mathrm{EU}_e(A_2)$. Which act does the evidential theory recommend?