PE3101P: Decision and Social Choice

Tutorial 2 Exercises

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Question 1

Let Cr satisfy the probability axioms on a sample space Ω. Prove the following results, identifying where you use the axioms. Results from earlier parts may be used in later parts.

(a) $\mathrm{Cr}(\varnothing) = 0$.

(b) If $A \subset B$, then $\mathrm{Cr}(B \setminus A) = \mathrm{Cr}(B) - \mathrm{Cr}(A)$. Hence show that $\mathrm{Cr}(A) \leq \mathrm{Cr}(B)$.

(c) For arbitrary events A, B,

$$\mathrm{Cr}(A \cup B) = \mathrm{Cr}(A) + \mathrm{Cr}(B) - \mathrm{Cr}(A \cap B).$$

(d) For arbitrary events A, B,

$$\mathrm{Cr}(A \cap B) \geq \mathrm{Cr}(A) + \mathrm{Cr}(B) - 1.$$

(e) If $\mathrm{Cr}(A) = 1$, then $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(B)$ for every event B.

(f) Let E be an event with $\mathrm{Cr}(E) > 0$, and define $\mathrm{Cr}_E(X) = \mathrm{Cr}(X \mid E)$. Prove that Cr_E satisfies:

(i) Non-negativity;

(ii) Normalisation;

(iii) Additivity.

In the Additivity proof, state which events are disjoint before applying Additivity to Cr.

Question 2

An agent must bring some notes to a meeting in exactly one of three ways: on paper, on a USB stick, or through an online account. Paper notes can always be read. Notes on the USB stick can be read exactly when the room's USB port works; notes in the online account can be read exactly when the room has internet access. Either condition of the USB port can occur with either condition of the internet connection. The only final outcomes we distinguish are that the notes can be read and that they cannot be read, since this is all the agent cares about.

(a) Identify the acts and the possible final outcomes.

(b) Give a set of mutually exclusive and jointly exhaustive states of nature which, together with each act, determine an outcome.

(c) Draw the resulting decision table and fill in every outcome.

Question 3

Events P and Q are mutually exclusive. An agent has credences

$$\mathrm{Cr}(P) = 0.30, \qquad \mathrm{Cr}(Q) = 0.25, \qquad \mathrm{Cr}(P \cup Q) = 0.35.$$

A bet pays its holder the amount shown below, after any purchase or sale price has been paid:

Bet	P occurs	Q occurs	Neither occurs
On P	\$20	\$0	\$0
On Q	\$0	\$20	\$0
On $P \cup Q$	\$20	\$20	\$0

Assume the betting interpretation: the agent buys a bet for any price below its stake times her credence in the event, and sells it for any price above that amount. A seller receives the price and pays the holder any payout that becomes due. The agent considers each offer on its own.

(a) Which probability axiom is violated, and what would it require here?

(b) Construct three trades, one involving each bet, which the agent will accept and which together give her a guaranteed loss. For each trade, state whether she buys or sells and the price. Calculate her final net gain or loss in each of the three states, including prices and payouts.

(c) Suppose instead that $\mathrm{Cr}(P \cup Q) = 0.8$, with the other credences unchanged. To arrange a guaranteed loss, would we need the agent to buy or sell a bet on $P \cup Q$? Explain why, taking into account the relationship between buying or selling two equal-sized bets on P and Q, and buying or selling a single bet of that size on $P \cup Q$.

Question 4

The gambles below use the same mutually exclusive and jointly exhaustive states. The agent's credence in each state is independent of which gamble is chosen. The letters a, b, c denote distinct final outcomes.

	s_1	s_2	s_3	s_4	s_5
Credence	0.10	0.15	0.25	0.20	0.30
G	a	a	b	c	c
H	b	b	a	c	c
J	a	b	a	c	c
K	c	c	b	a	c

Stochastic Equivalence requires indifference between gambles giving each final outcome the same probability.

(a) For each gamble, calculate the probabilities of a, b, c.

(b) Which pairs of distinct gambles must the agent be indifferent between if she satisfies Stochastic Equivalence?

(c) Construct another gamble M, different from K in at least one state, which has the same prospect as K. Specify the outcome of M in every state.

(d) Now change the state credences to $0.20, 0.15, 0.15, 0.20, 0.30$, respectively. Recalculate the prospects of G, H, J. Which pairs among these three must now be indifferent by Stochastic Equivalence?

Question 5

Recall that $X\ [p]\ Y$ assigns each final outcome o probability

$$(X\ [p]\ Y)(o) = pX(o) + (1-p)Y(o).$$

Suppose preferences satisfy Transitivity and Independence. For any prospects X, Y, Z, Transitivity says $X \succsim Y$ and $Y \succsim Z$ imply $X \succsim Z$. Independence says, for $0 < p < 1$,

$$X \succsim Y \iff X\ [p]\ Z \succsim Y\ [p]\ Z \iff Z\ [p]\ X \succsim Z\ [p]\ Y.$$

Here $\iff$ means "if and only if". Recall also $X\ [p]\ X = X$.

(a) Show that $X\ [0]\ Y = Y$ and $X\ [1]\ Y = X$, by showing that every final outcome has the same probability on both sides of each equation.

(b) Suppose $A \succsim B$ and $A \succsim C$. Prove that $A \succsim B\ [p]\ C$ for every $p \in [0, 1]$, using these steps:

(i) Use part (a) to establish the conclusion for $p = 0$ and $p = 1$.

(ii) For $0 < p < 1$, use Independence to establish

$$A\ [p]\ A \succsim A\ [p]\ C \quad \text{and} \quad A\ [p]\ C \succsim B\ [p]\ C.$$

(iii) Deduce the required conclusion for $0 < p < 1$.

(c) Suppose also $B \succsim D$ and $C \succsim D$. Prove that $A \succsim B\ [p]\ C \succsim D$ for every $p \in [0, 1]$.

Hint: For $0 < p < 1$, first compare $B\ [p]\ C$ with $D\ [p]\ C$, then compare $D\ [p]\ C$ with $D\ [p]\ D$. Cover $p = 0$ and $p = 1$ separately.

Question 6

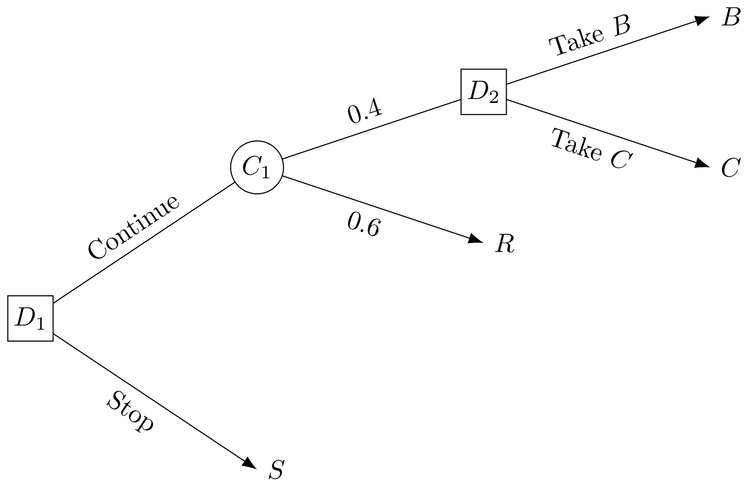
A reminder:

- Naive choice selects a currently preferred continuation, without anticipating a later departure from it. At each later decision node, the agent chooses afresh using the preferences held there.
- Sophisticated choice correctly predicts the later choice and evaluates its consequence using the utilities held at the current decision.

The whole tree is known in advance. Squares are decision nodes; circles are chance nodes.

The agent evaluates prospects by expected utility using the utilities held at the time of evaluation. The chance probabilities remain fixed.

	S	R	B	C
At D_1	5	2	12	1
At D_2	5	2	1	12



(a) List all complete plans and calculate the initial expected utility of each. A complete plan specifies a choice at both decision nodes, including any that it does not reach.

(b) Under naive choice, state the initial plan and the choices actually made at each reached decision node. Give the probabilities of the final outcomes.

(c) Under sophisticated choice, state the choice at each decision node if reached and the final outcome. Show the initial comparison that determines the choice at D_1.