PE3101P: Decision and Social Choice

Tutorial 2 Exercises — Solutions

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Question 1

Let Cr satisfy the probability axioms on a sample space Ω. Prove the following results, identifying where you use the axioms. Results from earlier parts may be used in later parts.

(a) $\mathrm{Cr}(\varnothing) = 0$.

Solution. Let X be any event. The events X and $\varnothing$ are disjoint, and $X \cup \varnothing = X$. By Additivity,

$$\mathrm{Cr}(X) = \mathrm{Cr}(X \cup \varnothing) = \mathrm{Cr}(X) + \mathrm{Cr}(\varnothing).$$

Subtracting $\mathrm{Cr}(X)$ from both sides gives $\mathrm{Cr}(\varnothing) = 0$. Normalisation is not needed.

In every sample space, both Ω and $\varnothing$ are events. Either can therefore be used for X: a proof using $\Omega \cup \varnothing$ or $\varnothing \cup \varnothing$ works in the same way.

(b) If $A \subset B$, then $\mathrm{Cr}(B \setminus A) = \mathrm{Cr}(B) - \mathrm{Cr}(A)$. Hence show that $\mathrm{Cr}(A) \leq \mathrm{Cr}(B)$.

Solution. The identity $\mathrm{Cr}(B \setminus A) = \mathrm{Cr}(B) - \mathrm{Cr}(A)$ is given, so we may assume it. Here is a proof of it for completeness. Since $A \subset B$,

$$B = A \cup (B \setminus A),$$

and the two events on the right are disjoint. Additivity gives

$$\mathrm{Cr}(B) = \mathrm{Cr}(A) + \mathrm{Cr}(B \setminus A).$$

Subtracting $\mathrm{Cr}(A)$ gives the stated identity:

$$\mathrm{Cr}(B \setminus A) = \mathrm{Cr}(B) - \mathrm{Cr}(A).$$

To prove the requested inequality, start from this identity and make $\mathrm{Cr}(B)$ the subject:

$$\mathrm{Cr}(B) = \mathrm{Cr}(A) + \mathrm{Cr}(B \setminus A).$$

Non-negativity gives $\mathrm{Cr}(B \setminus A) \geq 0$. Hence

$$\mathrm{Cr}(B) = \mathrm{Cr}(A) + \mathrm{Cr}(B \setminus A) \geq \mathrm{Cr}(A),$$

which is the required $\mathrm{Cr}(A) \leq \mathrm{Cr}(B)$.

(c) For arbitrary events A, B,

$$\mathrm{Cr}(A \cup B) = \mathrm{Cr}(A) + \mathrm{Cr}(B) - \mathrm{Cr}(A \cap B).$$

Solution. The union splits into three pairwise disjoint events:

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B).$$

The first contains only members of A outside B, the second only members of B outside A, and the third members of both. Applying Additivity twice gives

$$\mathrm{Cr}(A \cup B) = \mathrm{Cr}(A \setminus B) + \mathrm{Cr}(B \setminus A) + \mathrm{Cr}(A \cap B).$$

Also,

$$A = (A \setminus B) \cup (A \cap B).$$

These two events are disjoint, so Additivity gives

$$\mathrm{Cr}(A) = \mathrm{Cr}(A \setminus B) + \mathrm{Cr}(A \cap B).$$

Subtracting $\mathrm{Cr}(A \cap B)$ and making $\mathrm{Cr}(A \setminus B)$ the subject gives

$$\mathrm{Cr}(A \setminus B) = \mathrm{Cr}(A) - \mathrm{Cr}(A \cap B).$$

By the same argument,

$$\mathrm{Cr}(B \setminus A) = \mathrm{Cr}(B) - \mathrm{Cr}(A \cap B).$$

Substituting these expressions for $\mathrm{Cr}(A \setminus B)$ and $\mathrm{Cr}(B \setminus A)$ into the first credence equation gives

$$\begin{aligned}\mathrm{Cr}(A \cup B) &= \mathrm{Cr}(A) - \mathrm{Cr}(A \cap B) + \mathrm{Cr}(B) - \mathrm{Cr}(A \cap B) + \mathrm{Cr}(A \cap B)\\ &= \mathrm{Cr}(A) + \mathrm{Cr}(B) - \mathrm{Cr}(A \cap B),\end{aligned}$$

as required.

(d) For arbitrary events A, B,

$$\mathrm{Cr}(A \cap B) \geq \mathrm{Cr}(A) + \mathrm{Cr}(B) - 1.$$

Solution. Rearranging part (c), by adding $\mathrm{Cr}(A \cap B)$ and subtracting $\mathrm{Cr}(A \cup B)$, gives

$$\mathrm{Cr}(A \cap B) = \mathrm{Cr}(A) + \mathrm{Cr}(B) - \mathrm{Cr}(A \cup B).$$

Since $A \cup B \subset \Omega$, part (b) gives $\mathrm{Cr}(A \cup B) \leq \mathrm{Cr}(\Omega) = 1$, where the equality uses Normalisation. Multiplying this inequality by -1 reverses it:

$$-\mathrm{Cr}(A \cup B) \geq -1.$$

Adding $\mathrm{Cr}(A) + \mathrm{Cr}(B)$ and using the displayed identity gives

$$\mathrm{Cr}(A \cap B) \geq \mathrm{Cr}(A) + \mathrm{Cr}(B) - 1.$$

(e) If $\mathrm{Cr}(A) = 1$, then $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(B)$ for every event B.

Solution. We show that $\mathrm{Cr}(A \cap B)$ is both at least and at most $\mathrm{Cr}(B)$. By part (d) and $\mathrm{Cr}(A) = 1$,

$$\mathrm{Cr}(A \cap B) \geq 1 + \mathrm{Cr}(B) - 1 = \mathrm{Cr}(B).$$

Also $A \cap B \subset B$, so part (b) gives $\mathrm{Cr}(A \cap B) \leq \mathrm{Cr}(B)$. The two inequalities imply $\mathrm{Cr}(A \cap B) = \mathrm{Cr}(B)$.

(f) Let E be an event with $\mathrm{Cr}(E) > 0$, and define $\mathrm{Cr}_E(X) = \mathrm{Cr}(X \mid E)$. Prove that Cr_E satisfies:

(i) Non-negativity;

Solution. For every event X, the definition of conditional credence gives

$$\mathrm{Cr}_E(X) = \mathrm{Cr}(X \mid E) = \frac{\mathrm{Cr}(X \cap E)}{\mathrm{Cr}(E)}.$$

The numerator is non-negative by Non-negativity for Cr, and the denominator is strictly positive by the assumption on E. Therefore $\mathrm{Cr}_E(X) \geq 0$.

(ii) Normalisation;

Solution. We must show that $\mathrm{Cr}_E(\Omega) = 1$. Since $\Omega \cap E = E$,

$$\mathrm{Cr}_E(\Omega) = \frac{\mathrm{Cr}(\Omega \cap E)}{\mathrm{Cr}(E)} = \frac{\mathrm{Cr}(E)}{\mathrm{Cr}(E)} = 1.$$

The final division is valid because $\mathrm{Cr}(E) > 0$.

(iii) Additivity.

In the Additivity proof, state which events are disjoint before applying Additivity to Cr.

Solution. **(iii) Additivity.** Suppose X and Y are disjoint. Then $X \cap E$ and $Y \cap E$ are disjoint: a member of both would belong to $X \cap Y = \varnothing$. Also, by distribution of intersection over union,

$$(X \cup Y) \cap E = (X \cap E) \cup (Y \cap E).$$

Consequently,

$$\begin{aligned}
\mathrm{Cr}_E(X \cup Y) &= \frac{\mathrm{Cr}((X \cup Y) \cap E)}{\mathrm{Cr}(E)} && \text{(conditional credence)}\\
&= \frac{\mathrm{Cr}((X \cap E) \cup (Y \cap E))}{\mathrm{Cr}(E)} && \text{(set identity above)}\\
&= \frac{\mathrm{Cr}(X \cap E) + \mathrm{Cr}(Y \cap E)}{\mathrm{Cr}(E)} && \text{(Additivity for Cr)}\\
&= \frac{\mathrm{Cr}(X \cap E)}{\mathrm{Cr}(E)} + \frac{\mathrm{Cr}(Y \cap E)}{\mathrm{Cr}(E)} && \text{(division of a sum)}\\
&= \mathrm{Cr}_E(X) + \mathrm{Cr}_E(Y) && \text{(conditional credence).}
\end{aligned}$$

This is Additivity for Cr_E.

Question 2

An agent must bring some notes to a meeting in exactly one of three ways: on paper, on a USB stick, or through an online account. Paper notes can always be read. Notes on the USB stick can be read exactly when the room's USB port works; notes in the online account can be read exactly when the room has internet access. Either condition of the USB port can occur with either condition of the internet connection. The only final outcomes we distinguish are that the notes can be read and that they cannot be read, since this is all the agent cares about.

(a) Identify the acts and the possible final outcomes.

Solution. The three acts are bringing the notes on paper, on a USB stick, or through an online account. The two final outcomes are that the notes can be read and that they cannot be read.

(b) Give a set of mutually exclusive and jointly exhaustive states of nature which, together with each act, determine an outcome.

Solution. Use the four combinations of the two conditions:

State	USB port works	Internet available
s_1	Yes	Yes
s_2	Yes	No
s_3	No	Yes
s_4	No	No

Exactly one combination holds, so the states are mutually exclusive and jointly exhaustive. Each specifies both conditions, which is enough to determine whether the notes can be read for each of the three acts.

(c) Draw the resulting decision table and fill in every outcome.

Solution. Writing R for readable and U for unreadable, the table is

	s_1	s_2	s_3	s_4
Paper	R	R	R	R
USB	R	R	U	U
Online	R	U	R	U

Paper works in every state. The USB stick works in s_1, s_2, where the port works; the online account works in s_1, s_3, where internet is available.

Question 3

Events P and Q are mutually exclusive. An agent has credences

$$\mathrm{Cr}(P) = 0.30, \qquad \mathrm{Cr}(Q) = 0.25, \qquad \mathrm{Cr}(P \cup Q) = 0.35.$$

A bet pays its holder the amount shown below, after any purchase or sale price has been paid:

Bet	P occurs	Q occurs	Neither occurs
On P	\$20	\$0	\$0
On Q	\$0	\$20	\$0
On $P \cup Q$	\$20	\$20	\$0

Assume the betting interpretation: the agent buys a bet for any price below its stake times her credence in the event, and sells it for any price above that amount. A seller receives the price and pays the holder any payout that becomes due. The agent considers each offer on its own.

(a) Which probability axiom is violated, and what would it require here?

Solution. **Additivity is violated.** Because P and Q are mutually exclusive, it requires

$$\mathrm{Cr}(P \cup Q) = \mathrm{Cr}(P) + \mathrm{Cr}(Q) = 0.30 + 0.25 = 0.55,$$

whereas the agent's credence in the union is 0.35.

(b) Construct three trades, one involving each bet, which the agent will accept and which together give her a guaranteed loss. For each trade, state whether she buys or sells and the price. Calculate her final net gain or loss in each of the three states, including prices and payouts.

Solution. The agent's fair prices for the three bets are

$$20(0.30) = \$6, \qquad 20(0.25) = \$5, \qquad 20(0.35) = \$7.$$

Have her **buy the P bet for \$5**, **buy the Q bet for \$4**, and **sell the union bet for \$8**. Both purchases are below her respective fair prices, and the sale is above hers, so she accepts each offer.

Her initial net cash change is $-5 - 4 + 8 = -\$1$. The later payouts are:

Agent's cash change (\$)	P	Q	Neither
Prices paid/received	−1	−1	−1
Payout from P bet	20	0	0
Payout from Q bet	0	20	0
Payout owed on union bet	−20	−20	0
Total	−1	−1	−1

In state P, the total is $-1 + 20 + 0 - 20 = -1$; in state Q, it is $-1 + 0 + 20 - 20 = -1$; and if neither occurs, it is $-1 + 0 + 0 - 0 = -1$. Thus she loses \$1 in every state.

(c) Suppose instead that $\mathrm{Cr}(P \cup Q) = 0.8$, with the other credences unchanged. To arrange a guaranteed loss, would we need the agent to buy or sell a bet on $P \cup Q$? Explain why, taking into account the relationship between buying or selling two equal-sized bets on P and Q, and buying or selling a single bet of that size on $P \cup Q$.

Solution. She should **buy the union bet and sell the two component bets**. Since P and Q are disjoint, a \$20 union bet pays exactly the sum of a \$20 P bet and a \$20 Q bet: \$20 if either event occurs, and zero otherwise. Buying the union and selling the components therefore makes all later payouts cancel.

Her union fair price is now $20(0.8) = \$16$, while the two component fair prices sum to $6+5 = \$11$. She can therefore pay more for the union than she receives for the components while accepting each trade separately. For example, she buys the union for \$15, sells P for \$7, and sells Q for \$6. Each price is strictly favourable by her fair prices. Her net cash change is $-15+7+6 = -\$2$, and the payouts cancel in every state, so the loss is guaranteed.

Question 4

The gambles below use the same mutually exclusive and jointly exhaustive states. The agent's credence in each state is independent of which gamble is chosen. The letters a, b, c denote distinct final outcomes.

	s_1	s_2	s_3	s_4	s_5
Credence	0.10	0.15	0.25	0.20	0.30
G	a	a	b	c	c
H	b	b	a	c	c
J	a	b	a	c	c
K	c	c	b	a	c

Stochastic Equivalence requires indifference between gambles giving each final outcome the same probability.

(a) For each gamble, calculate the probabilities of a, b, c.

Solution. For each outcome, add the credences of the states in which the gamble gives it. This gives

	Probability of a	Probability of b	Probability of c
G	$0.10 + 0.15 = 0.25$	0.25	$0.20 + 0.30 = 0.50$
H	0.25	$0.10 + 0.15 = 0.25$	$0.20 + 0.30 = 0.50$
J	$0.10 + 0.25 = 0.35$	0.15	$0.20 + 0.30 = 0.50$
K	0.20	0.25	$0.10 + 0.15 + 0.30 = 0.55$

(b) Which pairs of distinct gambles must the agent be indifferent between if she satisfies Stochastic Equivalence?

Solution. **G and H.** Both give a, b, c probabilities $0.25, 0.25, 0.50$, respectively. Every other pair differs in at least one outcome probability, so Stochastic Equivalence alone requires no other indifference.

(c) Construct another gamble M, different from K in at least one state, which has the same prospect as K. Specify the outcome of M in every state.

Solution. One suitable gamble is

	s_1	s_2	s_3	s_4	s_5
M	b	b	c	a	c

It gives a probability 0.20, b probability $0.10 + 0.15 = 0.25$, and c probability $0.25 + 0.30 = 0.55$. These are K's outcome probabilities, although the gambles differ in states s_1, s_2, s_3. This is the only gamble different from K that has the same prospect as K with these states.

(d) Now change the state credences to $0.20, 0.15, 0.15, 0.20, 0.30$, respectively. Recalculate the prospects of G, H, J. Which pairs among these three must now be indifferent by Stochastic Equivalence?

Solution. Using the new state weights gives

	Probability of a	Probability of b	Probability of c
G	$0.20 + 0.15 = 0.35$	0.15	$0.20 + 0.30 = 0.50$
H	0.15	$0.20 + 0.15 = 0.35$	$0.20 + 0.30 = 0.50$
J	$0.20 + 0.15 = 0.35$	0.15	$0.20 + 0.30 = 0.50$

Now G **and** J have the same prospect and must be indifferent. Neither has the same prospect as H, so Stochastic Equivalence requires no further indifference.

Question 5

Recall that $X\ [p]\ Y$ assigns each final outcome o probability

$$(X\ [p]\ Y)(o) = pX(o) + (1 - p)Y(o).$$

Suppose preferences satisfy Transitivity and Independence. For any prospects X, Y, Z, Transitivity says $X \succsim Y$ and $Y \succsim Z$ imply $X \succsim Z$. Independence says, for $0 < p < 1$,

$$X \succsim Y \iff X\ [p]\ Z \succsim Y\ [p]\ Z \iff Z\ [p]\ X \succsim Z\ [p]\ Y.$$

Here $\iff$ means "if and only if". Recall also $X\ [p]\ X = X$.

(a) Show that $X\ [0]\ Y = Y$ and $X\ [1]\ Y = X$, by showing that every final outcome has the same probability on both sides of each equation.

Solution. Fix any final outcome o. Substituting $p = 0$ into the definition of a mixture gives

$$(X\ [0]\ Y)(o) = 0X(o) + (1 - 0)Y(o) = 0 + Y(o) = Y(o).$$

Thus $X\ [0]\ Y$ and Y assign every outcome the same probability, so they are the same prospect. Similarly,

$$(X\ [1]\ Y)(o) = 1X(o) + (1 - 1)Y(o) = X(o) + 0 = X(o).$$

This holds for every o, so $X\ [1]\ Y = X$.

(b) Suppose $A \succsim B$ and $A \succsim C$. Prove that $A \succsim B\ [p]\ C$ for every $p \in [0, 1]$, using these steps:

(i) Use part (a) to establish the conclusion for $p = 0$ and $p = 1$.

Solution. When $p = 0$, part (a) gives $B\ [0]\ C = C$, and $A \succsim C$ is assumed. Hence $A \succsim B\ [0]\ C$. When $p = 1$, it gives $B\ [1]\ C = B$, and the assumption $A \succsim B$ gives $A \succsim B\ [1]\ C$.

(ii) For $0 < p < 1$, use Independence to establish

$$A\ [p]\ A \succsim A\ [p]\ C \quad \text{and} \quad A\ [p]\ C \succsim B\ [p]\ C.$$

Solution. From $A \succsim C$, Independence lets us mix each with the same first component A, with first-component probability p. This gives

$$A\ [p]\ A \succsim A\ [p]\ C.$$

From $A \succsim B$, Independence lets us mix each with the same second component C, giving each of A, B probability p. Thus

$$A\ [p]\ C \succsim B\ [p]\ C.$$

Both uses of Independence are licensed because $0 < p < 1$.

(iii) Deduce the required conclusion for $0 < p < 1$.

Solution. Transitivity applied to the two comparisons in (ii) gives

$$A\ [p]\ A \succsim B\ [p]\ C.$$

Since $A\ [p]\ A = A$, this is $A \succsim B\ [p]\ C$. Together with (i), it proves the conclusion for every $p \in [0, 1]$.

(c) Suppose also $B \succsim D$ and $C \succsim D$. Prove that $A \succsim B\ [p]\ C \succsim D$ for every $p \in [0, 1]$.

Hint: For $0 < p < 1$, first compare $B\ [p]\ C$ with $D\ [p]\ C$, then compare $D\ [p]\ C$ with $D\ [p]\ D$. Cover $p = 0$ and $p = 1$ separately.

Solution. Part (b) already gives the upper bound $A \succsim B\ [p]\ C$. It remains to prove $B\ [p]\ C \succsim D$.

If $p = 0$, the mixture is C, and $C \succsim D$ is assumed. If $p = 1$, the mixture is B, and $B \succsim D$ is assumed.

Now let $0 < p < 1$. From $B \succsim D$, Independence with common second component C gives

$$B\ [p]\ C \succsim D\ [p]\ C.$$

From $C \succsim D$, Independence with common first component D gives

$$D\ [p]\ C \succsim D\ [p]\ D.$$

Transitivity therefore gives $B\ [p]\ C \succsim D\ [p]\ D = D$. Combining this with the upper bound and the two endpoint cases proves

$$A \succsim B\ [p]\ C \succsim D \qquad \text{for every } p \in [0, 1].$$

Question 6

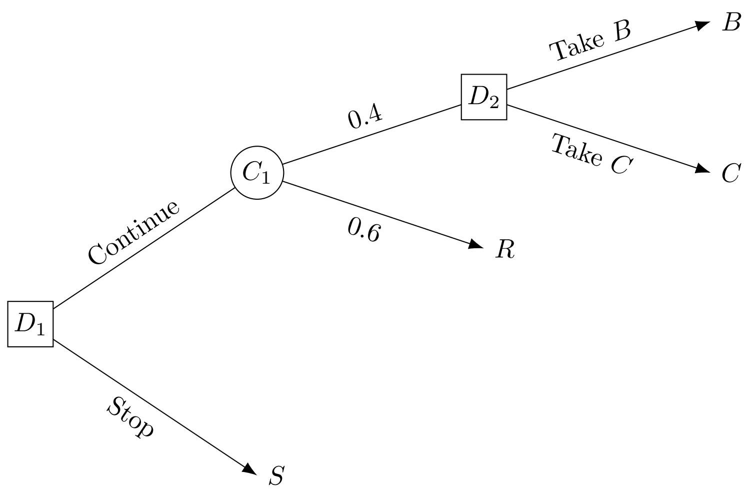
A reminder:

- Naive choice selects a currently preferred continuation, without anticipating a later departure from it. At each later decision node, the agent chooses afresh using the preferences held there.
- Sophisticated choice correctly predicts the later choice and evaluates its consequence using the utilities held at the current decision.

The whole tree is known in advance. Squares are decision nodes; circles are chance nodes.

The agent evaluates prospects by expected utility using the utilities held at the time of evaluation. The chance probabilities remain fixed.

	S	R	B	C
At D_1	5	2	12	1
At D_2	5	2	1	12



(a) List all complete plans and calculate the initial expected utility of each. A complete plan specifies a choice at both decision nodes, including any that it does not reach.

Solution. There are two choices at each of two decision nodes, hence $2 \times 2 = 4$ complete plans. Initial evaluation uses the D_1 utility row throughout, including for outcomes reached later:

Choice at D_1	Choice at D_2	Initial expected utility
Stop	Take B	$u_1(S) = 5$
Stop	Take C	$u_1(S) = 5$
Continue	Take B	$0.4(12) + 0.6(2) = 4.8 + 1.2 = 6$
Continue	Take C	$0.4(1) + 0.6(2) = 0.4 + 1.2 = 1.6$

The two stopping plans have the same result S: their different instructions for D_2 are never carried out.

(b) Under naive choice, state the initial plan and the choices actually made at each reached decision node. Give the probabilities of the final outcomes.

Solution. The initial plan is **Continue at D_1, then Take B at D_2**: its initial expected utility 6 exceeds 5 and 1.6.

The agent therefore continues at D_1. If D_2 is reached, she chooses afresh using that row: $u_2(C) = 12 > 1 = u_2(B)$, so she actually **takes** C. The chance node reaches D_2 with probability 0.4 and gives R with probability 0.6. Her final outcome probabilities are consequently

$$\mathrm{Cr}(C) = 0.4, \qquad \mathrm{Cr}(R) = 0.6, \qquad \mathrm{Cr}(S) = \mathrm{Cr}(B) = 0.$$

(c) Under sophisticated choice, state the choice at each decision node if reached and the final outcome. Show the initial comparison that determines the choice at D_1.

Solution. At D_2, if reached, the agent takes C because $12 > 1$ using the D_2 row. She predicts this at D_1, but evaluates its consequence using the D_1 row. Continuing therefore has expected utility

$$0.4u_1(C) + 0.6u_1(R) = 0.4(1) + 0.6(2) = 0.4 + 1.2 = 1.6.$$

Stopping gives $u_1(S) = 5$. Since $5 > 1.6$, she **stops at** D_1, and the final outcome is S **with certainty**. Her instruction at D_2 is Take C, but that node is not reached.