PE3101P: Decision and Social Choice

Tutorial 3 Exercises

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Throughout, an outcome is written as a list of numbers, one per person, in the order (person 1, person 2, ...), and a gamble is described by the outcome it gives in each state of nature. Unless stated otherwise, you may assume that the numbers in outcomes represent people's individual utilities, satisfying the Same Scale assumption.

Question 1

Utility functions u_1 and u_2 represent person 1's and person 2's preferences, respectively. They do not necessarily satisfy the Same Scale assumption. Their utilities for two options are:

	u_1	u_2
Option 1	3	10
Option 2	7	4

With these numbers, Option 1 has the greater sum of utilities.

(a) Suppose u_2 is replaced by $au_2 + c$, where $a > 0$, and u_1 is unchanged. For which values of a and c does Option 2 have the greater sum of utilities?

(b) What does (a) show about whether these two people's preferences settle which option has the greater sum of utilities?

Question 2

State the following principles:

(a) Statewise Dominance;

(b) Ex Ante Pareto.

Question 3

Two outcomes are *anonymously equivalent* if you can get from one to the other just by switching around who gets which number.

For each pair of outcomes, say whether they are anonymously equivalent. If they are, give a matching: for each person in the first outcome, name a person in the second outcome with the same number, using each person in the second outcome once (for example, "person 1 with person 3"). If they are not, explain why not.

(a) $(4, 1, 7)$ and $(7, 4, 1)$.

(b) $(3, 3, 6)$ and $(2, 5, 5)$.

(c) $(0, 4, 4, 8)$ and $(0, 0, 4, 8)$.

(d) $(1, 6, 1)$ and $(6, 1, 1)$.

Question 4

There are two people and a fair coin. Four gambles are:

	Heads	Tails
A	$(6, 2)$	$(6, 2)$
B	$(6, 2)$	$(2, 6)$
C	$(4, 4)$	$(4, 4)$
D	$(9, 1)$	$(1, 9)$

Throughout this question, use no comparisons between outcomes other than those given by Anonymity.

(a) Calculate each person's expected utility in each gamble.

(b) For each of the pairs A and B; B and C; C and D: what, if anything, does Statewise Dominance imply about how they compare? What, if anything, does Ex Ante Pareto imply?

(c) Is there a state of nature in which nobody is worse off under D than under C? Does your answer conflict with what Ex Ante Pareto implies about C and D? Explain.

(d) Using Transitivity and your answers to (b), how do A and D compare?

Question 5

There are three people and three equally likely states. C gives $(4, 4, 4)$ for sure.

(a) Construct a gamble that Ex Ante Pareto ranks above C, but in which, in every state, two people are worse off than under C.

(b) Can there be a gamble that Ex Ante Pareto ranks above C in which, in every state, *everyone* is worse off than under C? Explain.

Question 6

There are two people and a coin that lands Heads with probability $\frac{3}{4}$. Two gambles are:

	Heads	Tails
U	$(8, 0)$	$(8, 0)$
V	$(8, 0)$	$(0, 8)$

(a) Do Anonymity and Statewise Dominance still imply $U \sim V$? Explain.

(b) Calculate each person's expected utility in V. Which outcome, given for sure, do Expected Utility and Ex Ante Pareto imply is exactly as good as V?

(c) What do (a) and (b), with Transitivity, imply about how $(8, 0)$ compares with that outcome? Is this what Utilitarianism says?

Question 7

Recall that a lottery-equalisation of an outcome X is a gamble with as many equally likely states of nature as there are people, such that:

(i) in each state, the outcome is anonymously equivalent to X; and

(ii) across the states, each person occupies each position in the list X exactly once. (In each state, each person occupies a different position of X and gets the wellbeing of whoever is in that position in X.)

Construct a lottery-equalisation of each of the following outcomes, writing it in the blank table below the outcome.

(a) $(0, 2, 4, 6, 8)$.

	State 1	State 2	State 3	State 4	State 5
Person 1					
Person 2					
Person 3					
Person 4					
Person 5					

(b) $(1, 1, 1, 6, 6)$.

	State 1	State 2	State 3	State 4	State 5
Person 1					
Person 2					
Person 3					
Person 4					
Person 5					

(c) $(5, 0, 3, 0, 2)$.

	State 1	State 2	State 3	State 4	State 5
Person 1					
Person 2					
Person 3					
Person 4					
Person 5					

Question 8

Say that gamble P is *ex ante Pareto superior* to gamble Q if P is at least as good as Q for everyone and better than Q for someone.

$\mathsf{LE}(X)$ denotes a lottery-equalisation of X: a gamble with as many equally likely states as there are people such that (i) in each state, the outcome is anonymously equivalent to X; and (ii) across the states, each person occupies each position in the list X exactly once (in each state, each person occupies a different position of X and gets the wellbeing of whoever is in that position in X).

For each pair of outcomes X and Y below, answer the following.

(a) Is either of X and Y ex ante Pareto superior to the other?

(b) Is $\mathsf{LE}(X)$ ex ante Pareto superior to Y?

(c) Is $\mathsf{LE}(Y)$ ex ante Pareto superior to X?

(d) Is either of $\mathsf{LE}(X)$ and $\mathsf{LE}(Y)$ ex ante Pareto superior to the other?

		Person 1	Person 2	Person 3	Person 4
Pair 1	X	8	0	4	4
	Y	3	3	3	3
Pair 2	X	6	3	3	4
	Y	6	2	2	2
Pair 3	X	7	1	1	3
	Y	0	6	3	3
Pair 4	X	3	1	2	2
	Y	8	0	0	4

Question 9

In the proof below, you may appeal to any of the premises of Harsanyi's theorem: Anonymity, Statewise Dominance, Ex Ante Pareto, Expected Utility and Transitivity. (By Expected Utility, we mean the claim that the numbers are individual utilities satisfying Same Scale.)

Let $X = (6, 0, 3)$ and $Y = (1, 4, 1)$. $\mathsf{LE}(X)$ is a lottery-equalisation of X, and $\mathsf{E}(Y)$ is the equalisation of Y: the outcome in which everyone gets the average of Y's numbers. The following argument shows that $X \succ Y$. Fill in the gaps.

1. $X \sim \mathsf{LE}(X)$ _____ and _____
2. $\mathsf{LE}(X) \sim (3, 3, 3)$ _____ and _____
3. $X \sim (3, 3, 3)$ _____, from lines 1 and 2
4. $Y \sim \mathsf{E}(Y) = (____, ____, ____)$ as in lines 1–3, applied to Y
5. $(3, 3, 3) \succ \mathsf{E}(Y)$ _____ and _____
6. $X \succ Y$ _____, from lines ____, ____ and ____

Question 10

There are two people and a fair coin. Three gambles are:

	Heads	Tails
F	$(8, 0)$	$(8, 0)$
G	$(8, 0)$	$(0, 8)$
H	$(4, 4)$	$(4, 4)$

Each of the following rules gives every gamble a score, and ranks gambles by score: a higher score is better, and equal scores are equally good. (A rule that scores gambles like this is what we call a *social welfare function.*) Here e_1 and e_2 are person 1's and person 2's expected utilities in the gamble.

Rule A: $e_1 + e_2$.

Rule B: $e_1 + e_2$ plus the smaller of e_1 and e_2.

Rule C: the expectation, over the states, of the smaller of the two people's utilities in that state.

Rule D: $2e_1 + e_2$.

(Where the two numbers are equal, "the smaller" is either of them.)

(a) Calculate the score of F, G and H under each rule.

(b) Each of Rules B, C and D violates one of Anonymity, Statewise Dominance and Ex Ante Pareto (taking each person's expected utility to say what is better for them). For each rule, name the premise it violates, and give a pair of gambles or outcomes that shows it.

Question 11

In this question, w is a person's wellbeing and u is their utility.

(a) A person's utility is $u = \sqrt{w}$. A gamble gives them wellbeing 0 or wellbeing 16, with equal chances. Calculate the gamble's expected wellbeing and its expected utility. Which sure level of wellbeing is exactly as good for them as the gamble?

(b) Is this person's betterness relation risk averse, risk neutral or risk seeking:

(i) about wellbeing?

(ii) about utility?

Explain your answers.

(c) Three people have utility functions $u_1 = \sqrt{w}$, $u_2 = 2\sqrt{w}$ and $u_3 = w^2$. Answer the following for each of the pairs u_1 and u_2, and u_1 and u_3.

(i) Do the two functions satisfy Same Scale?

(ii) If not, can the second be stretched or shifted, that is, replaced by $au_2 + c$ (or $au_3 + c$) for some $a > 0$ and some c, so that together with u_1 it satisfies Same Scale? If so, give a and c. If not, explain why not.

(d) Suppose that everyone's utility is $u = \sqrt{w}$. Two outcomes give these levels of wellbeing:

$$P = (16, 0) \qquad Q = (4, 9).$$

Which has the greater sum of wellbeing? Which has the greater sum of utilities? If the premises of Harsanyi's theorem are true, which outcome is better?