PE3101P: Decision and Social Choice

Tutorial 3 Exercises — Solutions

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Throughout, an outcome is written as a list of numbers, one per person, in the order (person 1, person 2, ...), and a gamble is described by the outcome it gives in each state of nature. Unless stated otherwise, you may assume that the numbers in outcomes represent people's individual utilities, satisfying the Same Scale assumption.

Question 1

Utility functions u_1 and u_2 represent person 1's and person 2's preferences, respectively. They do not necessarily satisfy the Same Scale assumption. Their utilities for two options are:

	u_1	u_2
Option 1	3	10
Option 2	7	4

With these numbers, Option 1 has the greater sum of utilities.

(a) Suppose u_2 is replaced by $au_2 + c$, where $a > 0$, and u_1 is unchanged. For which values of a and c does Option 2 have the greater sum of utilities?

Solution: Person 2's new utilities are $10a + c$ for Option 1 and $4a + c$ for Option 2, so the sums of utilities are $3 + 10a + c$ for Option 1 and $7 + 4a + c$ for Option 2. Option 2 has the greater sum when

$$3 + 10a + c < 7 + 4a + c.$$

Subtracting c from both sides:

$$3 + 10a < 7 + 4a.$$

Subtracting $4a$ and 3 from both sides:

$$6a < 4,$$

that is, $a < \frac{2}{3}$. The c cancelled, so its value makes no difference. Option 2 has the greater sum exactly when $0 < a < \frac{2}{3}$, whatever c is.

(b) What does (a) show about whether these two people's preferences settle which option has the greater sum of utilities?

Solution: For any $a > 0$ and any c, $au_2 + c$ represents the same preferences as u_2. With u_2 itself ($a = 1$, $c = 0$), Option 1 has the greater sum: $3 + 10 = 13$ against $7 + 4 = 11$. With $\frac{1}{2}u_2$ ($a = \frac{1}{2}$, $c = 0$), Option 2 does: $3 + 5 = 8$ against $7 + 2 = 9$. Both represent person 2's preferences equally well. So the two people's preferences do not, by themselves, settle which option has the greater sum of utilities.

Question 2

State the following principles:

(a) Statewise Dominance;

Solution: Statewise Dominance: if gamble X is at least as good as gamble Y in every state of nature, then $X \succsim Y$. If X is also better than Y in some state of nature, then $X \succ Y$.

(b) Ex Ante Pareto.

Solution: Ex Ante Pareto: if gamble X is at least as good for everyone as gamble Y, then $X \succsim Y$. If X is also better for someone than Y, then $X \succ Y$.

Question 3

Two outcomes are *anonymously equivalent* if you can get from one to the other just by switching around who gets which number.

For each pair of outcomes, say whether they are anonymously equivalent. If they are, give a matching: for each person in the first outcome, name a person in the second outcome with the same number, using each person in the second outcome once (for example, "person 1 with person 3"). If they are not, explain why not.

(a) $(4, 1, 7)$ and $(7, 4, 1)$.

Solution: Yes. Apply the permutation (1 2 3): person 1's number in the first outcome (4) is held by person 2 in the second, person 2's (1) by person 3, and person 3's (7) by person 1. So person 1 is matched with person 2, person 2 with person 3, and person 3 with person 1.

(b) $(3, 3, 6)$ and $(2, 5, 5)$.

Solution: No. Switching who gets which number cannot change which numbers occur. The first outcome gives two people 3, and nobody in the second gets 3.

(c) $(0, 4, 4, 8)$ and $(0, 0, 4, 8)$.

Solution: No. Switching who gets which number cannot change how many people get each number. The first outcome gives 4 to two people and 0 to one; the second gives 4 to one person and 0 to two.

(d) $(1, 6, 1)$ and $(6, 1, 1)$.

Solution: Yes: (1 2). Persons 1 and 2 swap numbers, and person 3 keeps theirs. (1 3 2) also works.

Question 4

There are two people and a fair coin. Four gambles are:

	Heads	Tails
A	$(6, 2)$	$(6, 2)$
B	$(6, 2)$	$(2, 6)$
C	$(4, 4)$	$(4, 4)$
D	$(9, 1)$	$(1, 9)$

Throughout this question, use no comparisons between outcomes other than those given by Anonymity.

(a) Calculate each person's expected utility in each gamble.

Solution: Each state has probability $\frac{1}{2}$. Each person's expected utility is $\frac{1}{2}$ times their utility on Heads plus $\frac{1}{2}$ times their utility on Tails:

	Person 1	Person 2
A	$\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot 6 = 6$	$\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2$
B	$\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot 2 = 4$	$\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 6 = 4$
C	$\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 4 = 4$	$\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 4 = 4$
D	$\frac{1}{2} \cdot 9 + \frac{1}{2} \cdot 1 = 5$	$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 9 = 5$

(b) For each of the pairs A and B; B and C; C and D: what, if anything, does Statewise Dominance imply about how they compare? What, if anything, does Ex Ante Pareto imply?

Solution: **A and B.** Statewise Dominance: on Heads, both give $(6, 2)$; on Tails, they give $(6, 2)$ and $(2, 6)$, which are anonymously equivalent and so equally good by Anonymity. So in every state each gamble is at least as good as the other, and Statewise Dominance, applied both ways round, gives $A \sim B$. Ex Ante Pareto implies nothing: by Expected Utility, A is better for person 1 and B is better for person 2, so neither is at least as good as the other for everyone.

B and C. Statewise Dominance implies nothing: in each state B gives $(6, 2)$ or $(2, 6)$ and C gives $(4, 4)$, which are not anonymously equivalent, so we have no comparison between them. Ex Ante Pareto: each person's expected utility is 4 in both gambles, so by Expected Utility the two gambles are equally good for each person. Ex Ante Pareto, applied both ways round, gives $B \sim C$.

C and D. Statewise Dominance implies nothing: $(4, 4)$ is not anonymously equivalent to $(9, 1)$ or to $(1, 9)$. Ex Ante Pareto: each person's expected utility is 5 in D and 4 in C, so by Expected Utility D is better than C for each person. Ex Ante Pareto gives $D \succ C$.

(c) Is there a state of nature in which nobody is worse off under D than under C? Does your answer conflict with what Ex Ante Pareto implies about C and D? Explain.

Solution: No. On Heads, person 2 gets 1 under D and 4 under C; on Tails, person 1 gets 1 under D and 4 under C. So in each state, someone is worse off under D.

There is no conflict. Ex Ante Pareto compares the gambles overall for each person, not by comparing outcomes state by state. One gamble can be better than another for a person overall, and yet be worse for them in some state of nature.

For person 1, D gives a half chance of 9 and a half chance of 1, with expected utility 5, so by Expected Utility it is better for them than 4 for sure, though on Tails it gives them less. Likewise for person 2, who gets less under D on Heads. With more than one person, this means that each person can find one gamble better overall, and yet, whichever state of nature occurs, it is worse for somebody.

(d) Using Transitivity and your answers to (b), how do A and D compare?

Solution: From (b), $A \sim B$ and $B \sim C$, so $A \sim C$ by Transitivity (applied twice). Since $D \succ C$ and $C \succsim A$, Transitivity gives $D \succsim A$. And $A \succsim D$ is ruled out: with $C \succsim A$, Transitivity would give $C \succsim D$, contradicting $D \succ C$. So $D \succ A$.

Question 5

There are three people and three equally likely states. C gives $(4, 4, 4)$ for sure.

(a) Construct a gamble that Ex Ante Pareto ranks above C, but in which, in every state, two people are worse off than under C.

Solution: For example:

	State 1	State 2	State 3
Person 1	15	0	0
Person 2	0	15	0
Person 3	0	0	15

Each person gets 15 in one state and 0 in the other two, so each person's expected utility is $\frac{1}{3} \cdot 15 = 5$. In C, everyone's expected utility is 4. Since $5 > 4$, by Expected Utility the gamble is better than C for each person, and Ex Ante Pareto ranks it above C. In every state, two people get 0, less than the 4 they get under C. Any construction meeting the conditions is correct.

(b) Can there be a gamble that Ex Ante Pareto ranks above C in which, in every state, *everyone* is worse off than under C? Explain.

Solution: No. An expectation is a kind of weighted average. If every number being averaged is less than 4, the average is less than 4, however the weights are distributed. So if a person is worse off in every state than they are under C, their expected utility is less than 4. This holds generally: a person who is worse off in every state has a lower expected utility.

So a gamble that gives everyone less than 4 in every state is, by Expected Utility, worse than C for everyone. For Ex Ante Pareto to rank it above C, it would have to be at least as good as C for everyone. Instead, Ex Ante Pareto ranks C above it.

Question 6

There are two people and a coin that lands Heads with probability $\frac{3}{4}$. Two gambles are:

	Heads	Tails
U	$(8, 0)$	$(8, 0)$
V	$(8, 0)$	$(0, 8)$

(a) Do Anonymity and Statewise Dominance still imply $U \sim V$? Explain.

Solution: Yes. Neither premise mentions probabilities, so the biased coin makes no difference. On Heads, the two gambles give the same outcome; on Tails, they give $(8, 0)$ and $(0, 8)$, which are anonymously equivalent and so equally good by Anonymity. So in every state each gamble is at least as good as the other, and Statewise Dominance, applied both ways round, gives $U \sim V$.

(b) Calculate each person's expected utility in V. Which outcome, given for sure, do Expected Utility and Ex Ante Pareto imply is exactly as good as V?

Solution: Person 1: $\frac{3}{4} \cdot 8 + \frac{1}{4} \cdot 0 = 6$. Person 2: $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 8 = 2$.

The outcome is $(6, 2)$. Given for sure, it gives person 1 expected utility 6 and person 2 expected utility 2, the same as V. By Expected Utility, $(6, 2)$ for sure is exactly as good as V for each person, and Ex Ante Pareto, applied both ways round, gives $V \sim (6, 2)$.

(c) What do (a) and (b), with Transitivity, imply about how $(8, 0)$ compares with that outcome? Is this what Utilitarianism says?

Solution: U gives $(8, 0)$ for sure, so (a) says $(8, 0) \sim V$, and (b) says $V \sim (6, 2)$. By Transitivity, $(8, 0) \sim (6, 2)$.

Yes. The sums of utilities are $8 + 0 = 8$ and $6 + 2 = 8$, so Utilitarianism also says they are equally good. We could have known this without doing the sums: the principles that got us there (Anonymity, Statewise Dominance, Ex Ante Pareto and Transitivity) all follow from Utilitarianism, so anything they imply, Utilitarianism implies too.

Question 7

Recall that a lottery-equalisation of an outcome X is a gamble with as many equally likely states of nature as there are people, such that:

(i) in each state, the outcome is anonymously equivalent to X; and

(ii) across the states, each person occupies each position in the list X exactly once. (In each state, each person occupies a different position of X and gets the wellbeing of whoever is in that position in X.)

Construct a lottery-equalisation of each of the following outcomes, writing it in the blank table below the outcome.

(a) $(0, 2, 4, 6, 8)$.

(b) $(1, 1, 1, 6, 6)$.

(c) $(5, 0, 3, 0, 2)$.

Solution: One answer for each; any table meeting (i) and (ii) is correct. State 1 has the original outcome. In each later state, the numbers shift along by one person: person 1 gets what person 2 had in the state before, person 2 what person 3 had, and so on, with person 5 getting what person 1 had. So person 1's row is also the original outcome.

(a)

	State 1	State 2	State 3	State 4	State 5
Person 1	0	2	4	6	8
Person 2	2	4	6	8	0
Person 3	4	6	8	0	2
Person 4	6	8	0	2	4
Person 5	8	0	2	4	6

(b)

	State 1	State 2	State 3	State 4	State 5
Person 1	1	1	1	6	6
Person 2	1	1	6	6	1
Person 3	1	6	6	1	1
Person 4	6	6	1	1	1
Person 5	6	1	1	1	6

(c)

	State 1	State 2	State 3	State 4	State 5
Person 1	5	0	3	0	2
Person 2	0	3	0	2	5
Person 3	3	0	2	5	0
Person 4	0	2	5	0	3
Person 5	2	5	0	3	0

In each table, every column (state) is a rearrangement of the outcome, so (i) holds. Following any one person along their row, they take each position of the list once, so (ii) holds.

Question 8

Say that gamble P is *ex ante Pareto superior* to gamble Q if P is at least as good as Q for everyone and better than Q for someone.

$\mathsf{LE}(X)$ denotes a lottery-equalisation of X: a gamble with as many equally likely states as there are people such that (i) in each state, the outcome is anonymously equivalent to X; and (ii) across the states, each person occupies each position in the list X exactly once (in each state, each person occupies a different position of X and gets the wellbeing of whoever is in that position in X).

For each pair of outcomes X and Y below, answer the following.

(a) Is either of X and Y ex ante Pareto superior to the other?

(b) Is $\mathsf{LE}(X)$ ex ante Pareto superior to Y?

(c) Is $\mathsf{LE}(Y)$ ex ante Pareto superior to X?

(d) Is either of $\mathsf{LE}(X)$ and $\mathsf{LE}(Y)$ ex ante Pareto superior to the other?

		Person 1	Person 2	Person 3	Person 4
Pair 1	X	8	0	4	4
	Y	3	3	3	3
Pair 2	X	6	3	3	4
	Y	6	2	2	2
Pair 3	X	7	1	1	3
	Y	0	6	3	3
Pair 4	X	3	1	2	2
	Y	8	0	0	4

Solution: Throughout, by Expected Utility, what is better for a person is settled by their expected utility, and an outcome given for sure gives each person their number in it.

$\mathsf{LE}(X)$ has four equally likely states, and by (ii) each person gets each of X's four numbers exactly once across them. So each person's expected utility in $\mathsf{LE}(X)$ is $\frac{1}{4}$ times the sum of X's numbers: the average of X. So $\mathsf{LE}(X)$ gives everyone the same expected utility, X's average.

Pair 1. X's average is $\frac{1}{4}(8+0+4+4) = 4$; Y's is $\frac{1}{4}(3+3+3+3) = 3$.

(a) Neither. X is better than Y for persons 1, 3 and 4, but worse for person 2.

(b) Yes. $\mathsf{LE}(X)$ gives everyone expected utility 4, more than the 3 each person gets in Y.

(c) No. $\mathsf{LE}(Y)$ gives everyone expected utility 3, less than the 8 person 1 gets in X.

(d) $\mathsf{LE}(X)$ is ex ante Pareto superior to $\mathsf{LE}(Y)$: everyone gets 4 in $\mathsf{LE}(X)$ and 3 in $\mathsf{LE}(Y)$.

Pair 2. X's average is $\frac{1}{4}(6+3+3+4) = 4$; Y's is $\frac{1}{4}(6+2+2+2) = 3$.

(a) X is ex ante Pareto superior to Y. It is at least as good for everyone and better for persons 2, 3 and 4.

(b) No. $\mathsf{LE}(X)$ gives everyone 4, less than the 6 person 1 gets in Y.

(c) No. $\mathsf{LE}(Y)$ gives everyone 3, less than the 6 person 1 gets in X.

(d) $\mathsf{LE}(X)$ is ex ante Pareto superior to $\mathsf{LE}(Y)$: everyone gets 4 against 3.

Pair 3. X's average is $\frac{1}{4}(7+1+1+3) = 3$; Y's is $\frac{1}{4}(0+6+3+3) = 3$.

(a) Neither. X is better for person 1; Y is better for person 2.

(b) No. $\mathsf{LE}(X)$ gives everyone 3, less than the 6 person 2 gets in Y.

(c) No. $\mathsf{LE}(Y)$ gives everyone 3, less than the 7 person 1 gets in X.

(d) Neither. Both give everyone expected utility 3, so they are equally good for everyone and neither is better for anyone.

Pair 4. X's average is $\frac{1}{4}(3+1+2+2) = 2$; Y's is $\frac{1}{4}(8+0+0+4) = 3$.

(a) Neither. Y is better for person 1; X is better for person 2.

(b) No. $\mathsf{LE}(X)$ gives everyone 2, less than the 8 person 1 gets in Y.

(c) Yes. $\mathsf{LE}(Y)$ gives everyone 3. That is at least as much as each person gets in X, and more for persons 2, 3 and 4.

(d) $\mathsf{LE}(Y)$ is ex ante Pareto superior to $\mathsf{LE}(X)$: everyone gets 3 against 2.

Question 9

In the proof below, you may appeal to any of the premises of Harsanyi's theorem: Anonymity, Statewise Dominance, Ex Ante Pareto, Expected Utility and Transitivity. (By Expected Utility, we mean the claim that the numbers are individual utilities satisfying Same Scale.)

Let $X = (6, 0, 3)$ and $Y = (1, 4, 1)$. $\mathsf{LE}(X)$ is a lottery-equalisation of X, and $\mathsf{E}(Y)$ is the equalisation of Y: the outcome in which everyone gets the average of Y's numbers. The following argument shows that $X \succ Y$. Fill in the gaps.

1. $X \sim \mathsf{LE}(X)$	_____ and _____
2. $\mathsf{LE}(X) \sim (3, 3, 3)$	_____ and _____
3. $X \sim (3, 3, 3)$	_____, from lines 1 and 2
4. $Y \sim \mathsf{E}(Y) = (____, ____, ____)$	as in lines 1–3, applied to Y
5. $(3, 3, 3) \succ \mathsf{E}(Y)$	_____ and _____
6. $X \succ Y$	_____, from lines ____, ____ and ____

Solution: The completed argument:

1. $X \sim \mathsf{LE}(X)$	**Anonymity** and **Statewise Dominance**
2. $\mathsf{LE}(X) \sim (3,3,3)$	**Expected Utility** and **Ex Ante Pareto**
3. $X \sim (3,3,3)$	**Transitivity**, from lines 1 and 2
4. $Y \sim \mathsf{E}(Y) = (\mathbf{2},\mathbf{2},\mathbf{2})$	as in lines 1–3, applied to Y
5. $(3,3,3) \succ \mathsf{E}(Y)$	**Expected Utility** and **Ex Ante Pareto**
6. $X \succ Y$	**Transitivity**, from lines **3**, **5** and **4**

Line 1: in every state, $\mathsf{LE}(X)$ gives an outcome anonymously equivalent to X, which by Anonymity is exactly as good as X. So X for sure and $\mathsf{LE}(X)$ are equally good in every state, and Statewise Dominance, applied both ways round, gives $X \sim \mathsf{LE}(X)$.

Line 2: each person's expected utility in $\mathsf{LE}(X)$ is $\frac{1}{3}(6+0+3) = 3$, the same as in $(3,3,3)$ for sure. By Expected Utility, the two are equally good for each person, and Ex Ante Pareto, applied both ways round, gives $\mathsf{LE}(X) \sim (3,3,3)$.

Line 4: Y's average is $\frac{1}{3}(1+4+1) = 2$, so $\mathsf{E}(Y) = (2,2,2)$.

Line 5: everyone gets 3 in $(3,3,3)$ and 2 in $(2,2,2)$, so by Expected Utility $(3,3,3)$ is better for everyone, and Ex Ante Pareto gives $(3,3,3) \succ (2,2,2)$.

Line 6: lines 3, 5 and 4 give $X \sim (3,3,3) \succ (2,2,2) \sim Y$, so $X \succ Y$ by Transitivity.

Question 10

There are two people and a fair coin. Three gambles are:

	Heads	Tails
F	$(8,0)$	$(8,0)$
G	$(8,0)$	$(0,8)$
H	$(4,4)$	$(4,4)$

Each of the following rules gives every gamble a score, and ranks gambles by score: a higher score is better, and equal scores are equally good. (A rule that scores gambles like this is what we call a *social welfare function.*) Here e_1 and e_2 are person 1's and person 2's expected utilities in the gamble.

Rule A: $e_1 + e_2$.

Rule B: $e_1 + e_2$ plus the smaller of e_1 and e_2.

Rule C: the expectation, over the states, of the smaller of the two people's utilities in that state.

Rule D: $2e_1 + e_2$.

(Where the two numbers are equal, "the smaller" is either of them.)

(a) Calculate the score of F, G and H under each rule.

Solution: First, each person's expected utility in each gamble:

	e_1	e_2
F	$\frac{1}{2}\cdot 8+\frac{1}{2}\cdot 8=8$	$\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0=0$
G	$\frac{1}{2}\cdot 8+\frac{1}{2}\cdot 0=4$	$\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 8=4$
H	$\frac{1}{2}\cdot 4+\frac{1}{2}\cdot 4=4$	$\frac{1}{2}\cdot 4+\frac{1}{2}\cdot 4=4$

For Rule C, the smaller utility in each state is 0 under F (Heads and Tails), 0 under G (Heads and Tails), and 4 under H. The scores are:

	Rule A	Rule B	Rule C	Rule D
F	$8+0=8$	$8+0=8$	$\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0=0$	$2\cdot 8+0=16$
G	$4+4=8$	$8+4=12$	$\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0=0$	$2\cdot 4+4=12$
H	$4+4=8$	$8+4=12$	$\frac{1}{2}\cdot 4+\frac{1}{2}\cdot 4=4$	$2\cdot 4+4=12$

(b) Each of Rules B, C and D violates one of Anonymity, Statewise Dominance and Ex Ante Pareto (taking each person's expected utility to say what is better for them). For each rule, name the premise it violates, and give a pair of gambles or outcomes that shows it.

Solution: Rule B violates Statewise Dominance. F and G give the same outcome on Heads. On Tails, they give $(8,0)$ and $(0,8)$, which Rule B scores equally: $8+0=8$ each. So F and G are equally good in every state, and Statewise Dominance requires $F \sim G$. But Rule B scores G (12) above F (8).

Rule C violates Ex Ante Pareto. Each person's expected utility is 4 in both G and H, so G and H are equally good for everyone, and Ex Ante Pareto requires $G \sim H$. But Rule C scores H (4) above G (0). (Another pair: $(5,4)$ for sure scores 4 under Rule C, the same as $(4,4)$, though it is better for person 1 and no worse for person 2.)

Rule D violates Anonymity. $(8,0)$ and $(0,8)$ are anonymously equivalent, so Anonymity requires them to be equally good. But Rule D scores them $2\cdot 8+0=16$ and $2\cdot 0+8=8$.

Other pairs that show the same violation are correct.

Question 11

In this question, w is a person's wellbeing and u is their utility.

(a) A person's utility is $u=\sqrt{w}$. A gamble gives them wellbeing 0 or wellbeing 16, with equal chances. Calculate the gamble's expected wellbeing and its expected utility. Which sure level of wellbeing is exactly as good for them as the gamble?

Solution: Expected wellbeing:

$$\tfrac{1}{2}\cdot 0+\tfrac{1}{2}\cdot 16=8.$$

Expected utility:

$$\tfrac{1}{2}\sqrt{0}+\tfrac{1}{2}\sqrt{16}=\tfrac{1}{2}\cdot 0+\tfrac{1}{2}\cdot 4=2.$$

A sure level of wellbeing w is exactly as good as the gamble when its utility equals the gamble's expected utility:

$$\sqrt{w}=2,$$

so $w=4$.

(b) Is this person's betterness relation risk averse, risk neutral or risk seeking:

(i) about wellbeing?

(ii) about utility?

Explain your answers.

Solution: (i) Risk averse about wellbeing. By (a), wellbeing 4 for sure is exactly as good for them as the gamble, though the gamble gives more wellbeing on average (8).

(ii) Risk neutral about utility. By Expected Utility, a gamble is exactly as good for them as a sure thing whose utility is the gamble's expected utility. This follows from what utility is, not from anything about this person.

(c) Three people have utility functions $u_1 = \sqrt{w}$, $u_2 = 2\sqrt{w}$ and $u_3 = w^2$. Answer the following for each of the pairs u_1 and u_2, and u_1 and u_3.

(i) Do the two functions satisfy Same Scale?

(ii) If not, can the second be stretched or shifted, that is, replaced by $au_2 + c$ (or $au_3 + c$) for some $a > 0$ and some c, so that together with u_1 it satisfies Same Scale? If so, give a and c. If not, explain why not.

Solution: u_1 **and** u_2**.** (i) No. Same Scale requires that when two people get the same number, they are equally well off. Person 1 gets utility 2 at wellbeing 4 ($\sqrt{4} = 2$), and person 2 gets utility 2 at wellbeing 1 ($2\sqrt{1} = 2$): the same number, different wellbeing.

(ii) Yes: $a = \frac{1}{2}$ and $c = 0$. Then $\frac{1}{2}u_2 = \frac{1}{2} \cdot 2\sqrt{w} = \sqrt{w} = u_1$, so the two people get the same number exactly when they have the same wellbeing.

u_1 **and** u_3**.** (i) No. At wellbeing 4, person 3 gets utility $4^2 = 16$; person 1 gets utility 16 only at wellbeing 256 ($\sqrt{256} = 16$). The same number goes with different wellbeing.

(ii) No. Same Scale needs the two functions to agree at every level of wellbeing: $aw^2 + c = \sqrt{w}$ for every w. At $w = 0$ this gives

$$a \cdot 0 + c = 0, \quad \text{so } c = 0.$$

At $w = 1$ it then gives

$$a \cdot 1 + 0 = 1, \quad \text{so } a = 1.$$

But at $w = 4$ it would then give

$$1 \cdot 16 + 0 = 2,$$

which is false. So no a and c work. A stretch or a shift preserves the shape of the curve, but these utility functions have different shapes.

(d) Suppose that everyone's utility is $u = \sqrt{w}$. Two outcomes give these levels of wellbeing:

$$P = (16, 0) \qquad Q = (4, 9).$$

Which has the greater sum of wellbeing? Which has the greater sum of utilities? If the premises of Harsanyi's theorem are true, which outcome is better?

Solution: Sums of wellbeing: P gives $16 + 0 = 16$ and Q gives $4 + 9 = 13$, so P has the greater sum of wellbeing.

Sums of utilities: P gives $\sqrt{16} + \sqrt{0} = 4 + 0 = 4$ and Q gives $\sqrt{4} + \sqrt{9} = 2 + 3 = 5$, so Q has the greater sum of utilities.

If the premises of Harsanyi's theorem are true, Q is better: outcomes are ranked by their sums of utilities, not their sums of wellbeing. That is because the step from an outcome's lottery-equalisation to its equalisation is made by Ex Ante Pareto, and Ex Ante Pareto goes by what is better for each person, which, by Expected Utility, is their expected utility, not their expected wellbeing.