PE3101P: Decision and Social Choice

Tutorial 4 Exercises

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Throughout, an outcome is written as a list of numbers, one per person, in the order (person 1, person 2, ...), and a gamble is described by the outcome it gives in each state of nature. Unless stated otherwise, the numbers in outcomes are people's wellbeing levels.

Question 1

In this question, the number of people can differ between outcomes. Write an outcome by listing each person's wellbeing in turn, person 1 first, with Ω for a person who does not exist in that outcome. In (a)–(d), the only people who might exist are persons 1 to 5, so each outcome lists five people. For example, $(6, 6, \Omega, \Omega, \Omega)$ is the outcome in which persons 1 and 2 exist, each with wellbeing 6, and persons 3, 4 and 5 do not exist. A person whose wellbeing is 0 exists, with a neutral life; a person written Ω does not exist. So adding person 3, with wellbeing 0, to $(6, 6, \Omega, \Omega, \Omega)$ gives $(6, 6, 0, \Omega, \Omega)$.

Use only the following three principles.

> **Utilitarianism for a Fixed Group**: If exactly the same people exist in X and in Y, then $X \succsim Y$ iff the total wellbeing in X is at least the total wellbeing in Y.
>
> **Neutral Addition**: Adding people whose lives are neutral (wellbeing 0) leaves an outcome equally good.
>
> **Transitivity**: If $X \succsim Y$ and $Y \succsim Z$, then $X \succsim Z$.

Let $X = (2, 2, \Omega, \Omega, \Omega)$ and $Y = (\Omega, \Omega, 1, 1, 1)$.

(a) Is there an outcome in which only person 1 exists that these principles imply is equally good as X? If so, give one, with a chain of outcomes from X to it, written in this notation, saying which principle justifies each step.

(b) Is there an outcome in which only person 1 exists that these principles imply is equally good as Y? If so, give one, with a chain of outcomes from Y to it, written in this notation, saying which principle justifies each step.

(c) How do X and Y compare? Explain your answer.

(d) Consider the outcomes in which everyone who exists has wellbeing 1. Do these principles rank any of them above X? Explain your answer.

(e) Now suppose that there is no limit on how many people might exist, though every outcome still has finitely many people. X is still the outcome in which only persons 1 and 2 exist, each with wellbeing 2; in this part you need not write outcomes out in full. Is there a

positive wellbeing level so low that no outcome in which everyone who exists has that level is ranked above X by these principles? Give one, or explain why there is none.

Question 2

There are two people and a fair coin. The numbers in outcomes are the people's wellbeing levels: in the outcome $(196, 4)$, person 1 has 196 and person 2 has 4. Each person's betterness is risk neutral about wellbeing: what is better for a person is settled by their expected wellbeing. There are two options, each a gamble on the coin:

	Heads	Tails
Option 1	$(196, 4)$	$(4, 196)$
Option 2	$(81, 81)$	$(81, 81)$

Two versions of prioritarianism rank gambles, both with the priority weighting $f(w) = \sqrt{w}$. Each gives a gamble a score, and ranks gambles with higher scores higher.

Ex ante prioritarianism: work out each person's expected wellbeing. Then take the sum of f of those.

Ex post prioritarianism: work out the sum of f(wellbeing) in each state. Then take the expectation.

Recall the principles of

Statewise Dominance: If gamble X is at least as good as gamble Y in every state of nature, then $X \succsim Y$. If X is also better than Y in some state of nature, then $X \succ Y$.

Ex Ante Pareto: If gamble X is at least as good for everyone as gamble Y, then $X \succsim Y$. If X is also better for someone than Y, then $X \succ Y$.

In this question, outcomes are ranked as both versions rank outcomes given for sure: by the sum of f of the people's wellbeing levels.

(a) Fill in the scores.

	Option 1	Option 2
Ex ante prioritarianism		
Ex post prioritarianism		

(b) Does Statewise Dominance settle how Option 1 and Option 2 must be ranked? If so, how? Explain your answer.

(c) Does Ex Ante Pareto settle how Option 1 and Option 2 must be ranked? If so, how? Explain your answer.

(d) Write "yes" or "no" in each cell: does that version's ranking of Option 1 and Option 2 go against that principle?

	Statewise Dominance	Ex Ante Pareto
Ex ante prioritarianism		
Ex post prioritarianism		

Question 3

There are three people. Call an outcome **unequal** if not everyone in it has the same wellbeing.

For each of (a)–(c), give a gamble that fits the description by filling in the blank table, or explain why there is none.

(a) A coin that lands Heads with probability 0.75 is tossed. Everyone has the same expected wellbeing, and the outcomes on Heads and on Tails are both unequal.

	Heads	Tails
Person 1		
Person 2		
Person 3		

(b) A coin that lands Heads with probability 0.75 is tossed. Everyone has the same expected wellbeing, everyone has the same wellbeing on Heads, and the outcome on Tails is unequal.

	Heads	Tails
Person 1		
Person 2		
Person 3		

(c) There are three equally likely states. Everyone has the same expected wellbeing, everyone has the same wellbeing in State 1, and the outcomes in States 2 and 3 are both unequal.

	State 1	State 2	State 3
Person 1			
Person 2			
Person 3			

Question 4

There is only one person, and a fair coin. There are two options, each a gamble on the coin:

	Heads	Tails
Option 1	16	16
Option 2	0	256

The numbers are her wellbeing levels.

Consider three possibilities for her utility as a function u of her wellbeing w:

Case 1: $u(w) = w$.

Case 2: $u(w) = 2\sqrt{w} + 1$.

Case 3: $u(w) = w^{1/4}$, the fourth root of w.

In each case, what is better for her is settled by her expected utility. Ex post prioritarianism works out the sum of f(wellbeing) in each state, then takes the expectation; it uses the priority weighting $f(w) = \sqrt{w}$.

(a) Calculate the scores of Option 1 and Option 2 under ex post prioritarianism.

(b) Answer (i) and (ii) for each of Cases 1–3.

(i) Calculate her expected utility in Option 1 and in Option 2.

(ii) Does ex post prioritarianism's ranking of Option 1 and Option 2 go against Ex Ante Pareto? Explain your answer.

(c) For each of Cases 1–3, is there a pair of gambles on which ex post prioritarianism goes against Ex Ante Pareto? If so, give one. If not, explain why not.

Question 5

Recall that Gini egalitarianism ranks outcomes by total wellbeing × (1 − Gini coefficient); call this the outcome's **Gini value**. Recall too that the Gini value is a weighted sum of the people's wellbeing levels: the outcome with the larger weighted sum ranks higher. To find the weights, line people up from worst off to best off (people with equal wellbeing in any order). With n people, the weights are

$$\frac{2n-1}{n},\ \frac{2n-3}{n},\ \ldots,\ \frac{3}{n},\ \frac{1}{n}.$$

So with two people, the worse off has weight $\frac{3}{2}$ and the better off has weight $\frac{1}{2}$.

(a) (i) Does Gini egalitarianism rank (30,000, 30,000) above (29,000, 31,200)?

(ii) Let w be any wellbeing level above 1,000. Does Gini egalitarianism rank $(w,\ w)$ above $(w - 1{,}000,\ w + 1{,}200)$, whatever w is? Explain your answer.

(b) Is there a value of G for which Gini egalitarianism ranks $(20{,}000,\ 30{,}000 + G)$ above (30,000, 30,000)? If so, for which values?

(c) Now add some people, each with 100,000, to both outcomes in (a)(i). Is there a number of such people for which Gini egalitarianism ranks (29,000, 31,200, 100,000, ...) above (30,000, 30,000, 100,000, ...)? If so, what is the smallest? Explain your answer.

Question 6

Recall that Gini egalitarianism ranks outcomes by total wellbeing × (1 − Gini coefficient); call this the outcome's **Gini value**. Recall too that the Gini value is a weighted sum of the people's wellbeing levels: the outcome with the larger weighted sum ranks higher. To find the weights, line people up from worst off to best off (people with equal wellbeing in any order). With n people, the weights are

$$\frac{2n-1}{n},\ \frac{2n-3}{n},\ \ldots,\ \frac{3}{n},\ \frac{1}{n}.$$

So with two people the weights are $\frac{3}{2}$ and $\frac{1}{2}$, and with three they are $\frac{5}{3}$, 1 and $\frac{1}{3}$.

Each row of the table gives two outcomes, X and Y, for persons 1 and 2. Person 3 has the same wellbeing in X as in Y; call it z. It can be any positive number.

Answer (a) and (b) for each row. Write your answers in the table, and show your working below it.

(a) Which of X and Y does Gini egalitarianism rank higher when persons 1 and 2 are the only people?

(b) Is there a level z for person 3 at which Gini egalitarianism ranks X and Y the other way round from your answer to (a)? If so, give one. If not, write "none" and explain why there is none.

	X	Y	(a) Higher with persons 1 and 2 alone	(b) A level z that reverses it (or "none")
(i)	$(5,5)$	$(3,9)$		
(ii)	$(5,5)$	$(3,7)$		
(iii)	$(5,5)$	$(3,12)$		

Question 7

Eleven people, A to K, have wellbeing 30,000, 31,000, ..., 40,000: A has 30,000, B has 31,000, and so on up to K with 40,000. Each option gives one of them a gift of wellbeing, and nobody else changes: "2 to B" means that B gets 2 more units of wellbeing.

Each version of prioritarianism in this question has a multiplier m, a number with $0 < m < 1$. It ranks options by how much they add to the priority-weighted sum. Each unit of wellbeing adds the following amount, depending on where it falls:

Stretch of wellbeing	Each unit adds
30,000 to 31,000	1
31,000 to 32,000	m
32,000 to 33,000	m^2
33,000 to 34,000	m^3

And so on: multiplied by m with each further 1,000 up. For example, with $m = 0.8$, the option "3 to C" adds $3 \times 0.8^2 = 1.92$.

You may use: 0.9^{10} is about 0.35, and $0.25^{10} = \dfrac{1}{1{,}048{,}576}$.

(a) Take $m = 0.9$.

(i) How does this version rank 1 to A and 2 to B?

(ii) How does it rank 1 to A and 1,000 to K?

(b) Take $m = 0.25$.

(i) How does this version rank 1 to A and 2 to B?

(ii) How does it rank 1 to A and 1,000 to K?

(c) Is there a multiplier m for which the version ranks 1 to A above 2 to B, and also ranks 1,000 to K above 1 to A? If so, give one. If not, explain why not.