PE3101P: Decision and Social Choice

Tutorial 4 Exercises — Solutions

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Throughout, an outcome is written as a list of numbers, one per person, in the order (person 1, person 2, ...), and a gamble is described by the outcome it gives in each state of nature. Unless stated otherwise, the numbers in outcomes are people's wellbeing levels.

Question 1

In this question, the number of people can differ between outcomes. Write an outcome by listing each person's wellbeing in turn, person 1 first, with Ω for a person who does not exist in that outcome. In (a)–(d), the only people who might exist are persons 1 to 5, so each outcome lists five people. For example, $(6, 6, \Omega, \Omega, \Omega)$ is the outcome in which persons 1 and 2 exist, each with wellbeing 6, and persons 3, 4 and 5 do not exist. A person whose wellbeing is 0 exists, with a neutral life; a person written Ω does not exist. So adding person 3, with wellbeing 0, to $(6, 6, \Omega, \Omega, \Omega)$ gives $(6, 6, 0, \Omega, \Omega)$.

Use only the following three principles.

> **Utilitarianism for a Fixed Group**: If exactly the same people exist in X and in Y, then $X \succsim Y$ iff the total wellbeing in X is at least the total wellbeing in Y.
>
> **Neutral Addition**: Adding people whose lives are neutral (wellbeing 0) leaves an outcome equally good.
>
> **Transitivity**: If $X \succsim Y$ and $Y \succsim Z$, then $X \succsim Z$.

Let $X = (2, 2, \Omega, \Omega, \Omega)$ and $Y = (\Omega, \Omega, 1, 1, 1)$.

(a) Is there an outcome in which only person 1 exists that these principles imply is equally good as X? If so, give one, with a chain of outcomes from X to it, written in this notation, saying which principle justifies each step.

Solution: Yes: X'' below.

	Outcome	Total
X	$(2, 2, \Omega, \Omega, \Omega)$	4
X'	$(4, 0, \Omega, \Omega, \Omega)$	4
X''	$(4, \Omega, \Omega, \Omega, \Omega)$	4

$X \sim X'$	Utilitarianism for a Fixed Group
$X' \sim X''$	Neutral Addition
$X \sim X''$	Transitivity

(b) Is there an outcome in which only person 1 exists that these principles imply is equally good as Y? If so, give one, with a chain of outcomes from Y to it, written in this notation, saying which principle justifies each step.

Solution: Yes: Y''' below.

	Outcome	Total
Y	$(\Omega, \Omega, 1, 1, 1)$	3
Y'	$(0, \Omega, 1, 1, 1)$	3
Y''	$(3, \Omega, 0, 0, 0)$	3
Y'''	$(3, \Omega, \Omega, \Omega, \Omega)$	3

$Y \sim Y'$	Neutral Addition
$Y' \sim Y''$	Utilitarianism for a Fixed Group
$Y \sim Y''$	Transitivity
$Y'' \sim Y'''$	Neutral Addition
$Y \sim Y'''$	Transitivity

Other correct chains are possible (for example, adding person 2 at 0 as well).

(c) How do X and Y compare? Explain your answer.

Solution: $X \succ Y$. By Utilitarianism for a Fixed Group, $(4, \Omega, \Omega, \Omega, \Omega) \succ (3, \Omega, \Omega, \Omega, \Omega)$: the group is person 1 alone, and $4 > 3$. Putting this together with (a) and (b),

$$X \sim (4, \Omega, \Omega, \Omega, \Omega) \succ (3, \Omega, \Omega, \Omega, \Omega) \sim Y,$$

so $X \succ Y$ by Transitivity.

(d) Consider the outcomes in which everyone who exists has wellbeing 1. Do these principles rank any of them above X? Explain your answer.

Solution: Yes: Z below.

	Outcome	Total
Z	$(1, 1, 1, 1, 1)$	5
Z'	$(5, 0, 0, 0, 0)$	5
Z''	$(5, \Omega, \Omega, \Omega, \Omega)$	5
X'' (from (a))	$(4, \Omega, \Omega, \Omega, \Omega)$	4

$Z \sim Z'$	Utilitarianism for a Fixed Group
$Z' \sim Z''$	Neutral Addition
$Z \sim Z''$	Transitivity
$Z'' \succ X''$	Utilitarianism for a Fixed Group
$Z \succ X''$	Transitivity
$X'' \sim X$	(a)
$Z \succ X$	Transitivity

(e) Now suppose that there is no limit on how many people might exist, though every outcome still has finitely many people. X is still the outcome in which only persons 1 and 2 exist, each with wellbeing 2; in this part you need not write outcomes out in full. Is there a positive wellbeing level so low that no outcome in which everyone who exists has that level is ranked above X by these principles? Give one, or explain why there is none.

Solution: There is none. Take any positive level ε, and an outcome in which n people exist, each with wellbeing ε. As in (b), this outcome is equally good as person 1 alone at $n \times \varepsilon$: add person 1 at 0 first if person 1 does not exist (Neutral Addition), give person 1 the whole total (Utilitarianism for a Fixed Group), then remove everyone else, who now has 0 (Neutral Addition). By (a), X is equally good as person 1 alone at 4. So if $n \times \varepsilon > 4$, Utilitarianism for a Fixed Group and Transitivity rank the outcome above X, as in (d). Whatever ε is, we can choose n bigger than $\frac{4}{\varepsilon}$, which makes $n \times \varepsilon > 4$. For example, an outcome with 41 or more people, each at 0.1, is ranked above X.

Question 2

There are two people and a fair coin. The numbers in outcomes are the people's wellbeing levels: in the outcome $(196, 4)$, person 1 has 196 and person 2 has 4. Each person's betterness is risk neutral about wellbeing: what is better for a person is settled by their expected wellbeing. There are two options, each a gamble on the coin:

	Heads	Tails
Option 1	$(196, 4)$	$(4, 196)$
Option 2	$(81, 81)$	$(81, 81)$

Two versions of prioritarianism rank gambles, both with the priority weighting $f(w) = \sqrt{w}$. Each gives a gamble a score, and ranks gambles with higher scores higher.

Ex ante prioritarianism: work out each person's expected wellbeing. Then take the sum of f of those.

Ex post prioritarianism: work out the sum of f(wellbeing) in each state. Then take the expectation.

Recall the principles of

Statewise Dominance: If gamble X is at least as good as gamble Y in every state of nature, then $X \succsim Y$. If X is also better than Y in some state of nature, then $X \succ Y$.

Ex Ante Pareto: If gamble X is at least as good for everyone as gamble Y, then $X \succsim Y$. If X is also better for someone than Y, then $X \succ Y$.

In this question, outcomes are ranked as both versions rank outcomes given for sure: by the sum of f of the people's wellbeing levels.

(a) Fill in the scores.

Solution: Ex ante. In Option 1, person 1's expected wellbeing is

$$\tfrac{1}{2} \cdot 196 + \tfrac{1}{2} \cdot 4 = 98 + 2 = 100,$$

and person 2's is

$$\tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot 196 = 2 + 98 = 100,$$

so the score is $\sqrt{100}+\sqrt{100}=10+10=20$. In Option 2, each person's expected wellbeing is 81, so the score is $\sqrt{81}+\sqrt{81}=9+9=18$.

Ex post. In Option 1, the sum of f(wellbeing) is $\sqrt{196}+\sqrt{4}=14+2=16$ on Heads, and $\sqrt{4}+\sqrt{196}=2+14=16$ on Tails, so the score is $\frac{1}{2}\cdot 16+\frac{1}{2}\cdot 16=16$. In Option 2, the sum is $9+9=18$ in each state, so the score is 18.

	Option 1	Option 2
Ex ante prioritarianism	20	18
Ex post prioritarianism	16	18

(b) Does Statewise Dominance settle how Option 1 and Option 2 must be ranked? If so, how? Explain your answer.

Solution: Yes: Option 2 $\succ$ Option 1. On Heads, Option 2 gives $(81, 81)$, whose sum of f is $9+9=18$, and Option 1 gives $(196, 4)$, whose sum of f is $14+2=16$. So Option 2 is better on Heads. On Tails, Option 1 gives $(4, 196)$, whose sum of f is also 16, so Option 2 is better on Tails too. Option 2 is at least as good in every state, and better in some, so Statewise Dominance gives Option 2 $\succ$ Option 1.

(c) Does Ex Ante Pareto settle how Option 1 and Option 2 must be ranked? If so, how? Explain your answer.

Solution: Yes: Option 1 $\succ$ Option 2. By (a), each person's expected wellbeing is 100 in Option 1 and 81 in Option 2. What is better for each person is settled by their expected wellbeing, so Option 1 is better for each of them, and Ex Ante Pareto gives Option 1 $\succ$ Option 2.

(d) Write "yes" or "no" in each cell: does that version's ranking of Option 1 and Option 2 go against that principle?

Solution:

	Statewise Dominance	Ex Ante Pareto
Ex ante prioritarianism	yes	no
Ex post prioritarianism	no	yes

Ex ante prioritarianism ranks Option 1 higher (20 against 18): against Statewise Dominance, by (b), and in line with Ex Ante Pareto, by (c). Ex post prioritarianism ranks Option 2 higher (18 against 16): in line with Statewise Dominance, and against Ex Ante Pareto.

Question 3

There are three people. Call an outcome **unequal** if not everyone in it has the same wellbeing.

For each of (a)–(c), give a gamble that fits the description by filling in the blank table, or explain why there is none.

(a) A coin that lands Heads with probability 0.75 is tossed. Everyone has the same expected wellbeing, and the outcomes on Heads and on Tails are both unequal.

Solution: For example:

	Heads	Tails
Person 1	1	1
Person 2	1	1
Person 3	0	4

Both outcomes, $(1, 1, 0)$ and $(1, 1, 4)$, are unequal. Persons 1 and 2 each have expected wellbeing $0.75 \cdot 1 + 0.25 \cdot 1 = 1$, and person 3 has $0.75 \cdot 0 + 0.25 \cdot 4 = 0 + 1 = 1$. Any gamble that fits the description is correct.

(b) A coin that lands Heads with probability 0.75 is tossed. Everyone has the same expected wellbeing, everyone has the same wellbeing on Heads, and the outcome on Tails is unequal.

Solution: There is none. Say everyone has wellbeing a on Heads, and person i has wellbeing w_i on Tails. Then person i's expected wellbeing is

$$0.75a + 0.25w_i.$$

For this to be the same for all three people, $0.25w_1 = 0.25w_2 = 0.25w_3$ (subtract $0.75a$ from each), so $w_1 = w_2 = w_3$. Then everyone has the same wellbeing on Tails too, so the outcome on Tails is not unequal.

(c) There are three equally likely states. Everyone has the same expected wellbeing, everyone has the same wellbeing in State 1, and the outcomes in States 2 and 3 are both unequal.

Solution: For example:

	State 1	State 2	State 3
Person 1	0	3	0
Person 2	0	0	3
Person 3	0	0	3

States 2 and 3 give $(3, 0, 0)$ and $(0, 3, 3)$, both unequal. Person 1 has expected wellbeing $\frac{1}{3}(0 + 3 + 0) = 1$, and persons 2 and 3 each have $\frac{1}{3}(0 + 0 + 3) = 1$. Any gamble that fits the description is correct.

Question 4

There is only one person, and a fair coin. There are two options, each a gamble on the coin:

	Heads	Tails
Option 1	16	16
Option 2	0	256

The numbers are her wellbeing levels.

Consider three possibilities for her utility as a function u of her wellbeing w:

Case 1: $u(w) = w$.

Case 2: $u(w) = 2\sqrt{w} + 1$.

Case 3: $u(w) = w^{1/4}$, the fourth root of w.

In each case, what is better for her is settled by her expected utility. Ex post prioritarianism works out the sum of f(wellbeing) in each state, then takes the expectation; it uses the priority weighting $f(w) = \sqrt{w}$.

(a) Calculate the scores of Option 1 and Option 2 under ex post prioritarianism.

Solution: With one person, the sum of f(wellbeing) in a state is just $\sqrt{w}$ for her wellbeing w in that state.

Option 1: $\sqrt{16} = 4$ in each state, so the score is 4.

Option 2: $\frac{1}{2} \cdot \sqrt{0} + \frac{1}{2} \cdot \sqrt{256} = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 16 = 8$.

(b) Answer (i) and (ii) for each of Cases 1–3.

(i) Calculate her expected utility in Option 1 and in Option 2.

(ii) Does ex post prioritarianism's ranking of Option 1 and Option 2 go against Ex Ante Pareto? Explain your answer.

Solution: Case 1. (i) Her utility is her wellbeing, so her expected utility is her expected wellbeing. Option 1: 16. Option 2: $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 256 = 128$.

(ii) No. Her expected utility is higher in Option 2 (128 against 16), so Option 2 is better for her, and Ex Ante Pareto requires Option 2 $\succ$ Option 1, which is how ex post prioritarianism ranks them (8 against 4, by (a)).

Case 2. (i) Option 1: $2\sqrt{16} + 1 = 2 \cdot 4 + 1 = 9$ in each state, so 9. Option 2:

$$\tfrac{1}{2}(2\sqrt{0} + 1) + \tfrac{1}{2}(2\sqrt{256} + 1) = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 33 = 17.$$

(ii) No. Her expected utility is higher in Option 2 (17 against 9), so Ex Ante Pareto requires Option 2 $\succ$ Option 1, which, as before, is how ex post prioritarianism ranks them.

Case 3. (i) Option 1: $16^{1/4} = 2$ in each state, so 2. Option 2:

$$\tfrac{1}{2} \cdot 0^{1/4} + \tfrac{1}{2} \cdot 256^{1/4} = \tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 4 = 2.$$

(ii) Yes. Her expected utility is 2 in both, so Ex Ante Pareto, applied both ways round, requires the two options to be ranked equally good; but, as before, ex post prioritarianism ranks Option 2 above Option 1.

(c) For each of Cases 1–3, is there a pair of gambles on which ex post prioritarianism goes against Ex Ante Pareto? If so, give one. If not, explain why not.

Solution: Case 1: yes. For example, take 1 for sure against a fair coin giving her 0 on Heads and 4 on Tails. Her expected utility is 1 in the sure thing and $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 4 = 2$ in the gamble, so the gamble is better for her, and Ex Ante Pareto requires it to be ranked higher. But ex post prioritarianism gives both the same score: $\sqrt{1} = 1$ for the sure thing, and

$$\tfrac{1}{2} \cdot \sqrt{0} + \tfrac{1}{2} \cdot \sqrt{4} = \tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot 2 = 1$$

for the gamble. Other pairs are correct too.

Case 2: no. Take any gamble, in which her wellbeing is a_1 in state 1, …, a_k in state k, with probabilities $p_1, \ldots, p_k$. Her expected utility is

$$p_1(2\sqrt{a_1} + 1) + \cdots + p_k(2\sqrt{a_k} + 1) = 2\big(p_1\sqrt{a_1} + \cdots + p_k\sqrt{a_k}\big) + (p_1 + \cdots + p_k).$$

The probabilities add up to 1, and with one person, $p_1\sqrt{a_1} + \cdots + p_k\sqrt{a_k}$ is the gamble's ex post score. So her expected utility is $2 \times$ (the ex post score) $+ 1$. Multiplying by 2 and adding 1 keeps numbers in the same order. So whenever one gamble is at least as good for her as another (so its expected utility is at least as high), its ex post score is at least as high, and whenever it is better for her, its score is higher. Ex post prioritarianism ranks gambles exactly as Ex Ante Pareto requires.

Case 3: yes. Option 1 and Option 2 themselves, by (b).

In general, with one person, ex post prioritarianism ranks gambles by the expectation of $f(w)$, and Ex Ante Pareto requires them to be ranked by her expected utility, the expectation of $u(w)$. So the question is whether f and u represent the same preferences over gambles. We know from the first half of the course that they do exactly when one is a positive affine transformation of the other: $u(w) = \alpha f(w) + \beta$ for every w, where α is a positive number and β is any real number. In Case 2, $u(w) = 2f(w) + 1$.

Question 5

Recall that Gini egalitarianism ranks outcomes by total wellbeing $\times$ (1 − Gini coefficient); call this the outcome's **Gini value**. Recall too that the Gini value is a weighted sum of the people's wellbeing levels: the outcome with the larger weighted sum ranks higher. To find the weights, line people up from worst off to best off (people with equal wellbeing in any order). With n people, the weights are

$$\frac{2n-1}{n},\ \frac{2n-3}{n},\ \ldots,\ \frac{3}{n},\ \frac{1}{n}.$$

So with two people, the worse off has weight $\frac{3}{2}$ and the better off has weight $\frac{1}{2}$.

(a) (i) Does Gini egalitarianism rank (30,000, 30,000) above (29,000, 31,200)?

(ii) Let w be any wellbeing level above 1,000. Does Gini egalitarianism rank $(w,\ w)$ above $(w - 1{,}000,\ w + 1{,}200)$, whatever w is? Explain your answer.

Solution: Each part of this question can be answered by calculating the Gini values directly, and that is perfectly fine. But each change here is a trade-off between two people, and comparing the ratio of their weights is faster. When a change moves wellbeing between just two people, and nobody changes place in the line, everyone else's term in the weighted sum stays the same. So if the worse off of the two has weight a and the better off has weight

b, a loss to the worse off is outweighed by a gain to the better off exactly when the gain is more than $\frac{a}{b}$ times the loss.

(i) Yes. Going from (30,000, 30,000) to (29,000, 31,200), person 1 loses 1,000 and person 2 gains 1,200. Person 1 is the worse off, with weight $\frac{3}{2}$, and person 2 the better off, with weight $\frac{1}{2}$: a ratio of 3. So the gain would have to be more than $3 \times 1{,}000 = 3{,}000$. It is 1,200, so the equal outcome ranks higher.

(ii) Yes. Going from (w, w) to $(w - 1{,}000, w + 1{,}200)$ is the same trade-off, whatever w is: person 1, the worse off, loses 1,000 at weight $\frac{3}{2}$, and person 2 gains 1,200 at weight $\frac{1}{2}$. So, as in (i), the gain would have to be more than 3,000, and (w, w) ranks higher at every level w.

(b) Is there a value of G for which Gini egalitarianism ranks (20,000, 30,000 + G) above (30,000, 30,000)? If so, for which values?

Solution: Yes: exactly when $G > 30{,}000$. Going from (30,000, 30,000) to (20,000, 30,000 + G), person 1 loses 10,000 and person 2's wellbeing changes by G.

If $G \geq -10{,}000$, person 1 is still the worse off, so this is a trade-off with the same weights as in (a): a ratio of 3. So (20,000, 30,000 + G) ranks higher exactly when person 2's gain G is more than $3 \times 10{,}000 = 30{,}000$. At $G = 30{,}000$ the two outcomes are equally good.

If $G < -10{,}000$, person 2 is the worse off, so we calculate directly. (30,000, 30,000) has Gini value $\frac{3}{2} \cdot 30{,}000 + \frac{1}{2} \cdot 30{,}000 = 60{,}000$, and (20,000, 30,000 + G) has

$$\tfrac{3}{2}(30{,}000 + G) + \tfrac{1}{2} \cdot 20{,}000 = 45{,}000 + \tfrac{3}{2}G + 10{,}000 = 55{,}000 + \tfrac{3}{2}G.$$

Since $G < -10{,}000$, this is less than $55{,}000 - 15{,}000 = 40{,}000$, below 60,000.

(c) Now add some people, each with 100,000, to both outcomes in (a)(i). Is there a number of such people for which Gini egalitarianism ranks (29,000, 31,200, 100,000, ...) above (30,000, 30,000, 100,000, ...)? If so, what is the smallest? Explain your answer.

Solution: Yes: the smallest number is five.

Going from the equal outcome to the unequal one is the same trade-off as in (a)(i): person 1 loses 1,000 and person 2 gains 1,200, and the added people, who are above both of them, are unaffected. Say k people are added, so there are $n = k + 2$ people. Persons 1 and 2 are the two worst off, so their weights are $\frac{2n-1}{n}$ and $\frac{2n-3}{n}$, a ratio of

$$r = \frac{2n - 1}{2n - 3}.$$

So the unequal outcome ranks higher exactly when $1{,}200 > r \times 1{,}000$, that is, when $r < 1.2$. We want the smallest number of added people that brings r below 1.2:

Added people k	1	2	3	4	5
People n	3	4	5	6	7
r	$\frac{5}{3} \approx 1.67$	$\frac{7}{5} = 1.4$	$\frac{9}{7} \approx 1.29$	$\frac{11}{9} \approx 1.22$	$\frac{13}{11} \approx 1.18$

With four or fewer added people, r is above 1.2, so the equal outcome still ranks higher; with five, r is below 1.2, so the unequal outcome ranks higher.

Question 6

Recall that Gini egalitarianism ranks outcomes by total wellbeing × (1 − Gini coefficient); call this the outcome's **Gini value**. Recall too that the Gini value is a weighted sum of the people's wellbeing levels: the outcome with the larger weighted sum ranks higher. To find the weights, line people up from worst off to best off (people with equal wellbeing in any order). With n people, the weights are

$$\frac{2n-1}{n},\ \frac{2n-3}{n},\ \ldots,\ \frac{3}{n},\ \frac{1}{n}.$$

So with two people the weights are $\frac{3}{2}$ and $\frac{1}{2}$, and with three they are $\frac{5}{3}$, 1 and $\frac{1}{3}$.

Each row of the table gives two outcomes, X and Y, for persons 1 and 2. Person 3 has the same wellbeing in X as in Y; call it z. It can be any positive number.

Answer (a) and (b) for each row. Write your answers in the table, and show your working below it.

(a) Which of X and Y does Gini egalitarianism rank higher when persons 1 and 2 are the only people?

(b) Is there a level z for person 3 at which Gini egalitarianism ranks X and Y the other way round from your answer to (a)? If so, give one. If not, write "none" and explain why there is none.

Solution:

	X	Y	(a) Higher with persons 1 and 2 alone	(b) A level z that reverses it (or "none")
(i)	$(5,5)$	$(3,9)$	X	e.g. $z = 10$
(ii)	$(5,5)$	$(3,7)$	X	none
(iii)	$(5,5)$	$(3,12)$	Y	e.g. $z = 5$

Multiplying every weight by the same positive number does not change which outcome has the larger weighted sum. So below, the two-person weights are doubled, to 3 and 1, and the three-person weights are tripled, to 5, 3 and 1, always from worst off to best off.

Each row can be answered by calculating these weighted sums directly, as below. Each row is also a trade-off between persons 1 and 2, as in Question 5, and comparing the ratio of their weights is faster: going from X to Y, person 1 loses 2 and person 2 gains something, while person 3 is unaffected. So, when nobody changes place in the line, Y ranks higher exactly when person 2's gain is more than the ratio of the two weights times person 1's loss of 2.

Row (i). (a) X: $3 \cdot 5 + 1 \cdot 5 = 20$. Y: $3 \cdot 3 + 1 \cdot 9 = 18$. So X ranks higher. By the ratio: the weights are 3 and 1, a ratio of 3, so person 2's gain would have to be more than $3 \times 2 = 6$. It is 4.

(b) Take $z = 10$. Then $X = (5, 5, 10)$ gives $5 \cdot 5 + 3 \cdot 5 + 1 \cdot 10 = 25 + 15 + 10 = 50$, and $Y = (3, 9, 10)$ gives $5 \cdot 3 + 3 \cdot 9 + 1 \cdot 10 = 15 + 27 + 10 = 52$. So Y ranks higher. By the ratio: with person 3 above both of them, persons 1 and 2 have the weights 5 and 3, a ratio of $\frac{5}{3}$, so the gain only has to be more than $\frac{5}{3} \times 2 \approx 3.3$. It is 4.

Row (ii). (a) X: 20, as in row (i). Y: $3 \cdot 3 + 1 \cdot 7 = 16$. So X ranks higher. By the ratio: as in row (i), the gain would have to be more than 6. It is 2.

(b) None. X comes from Y by a pure transfer of 2 units of wellbeing from person 2, who is better off, to person 1, who is worse off: person 2 goes from 7 to 5, person 1 from 3 to 5, and person 3

is unaffected. Afterwards both have 5, so the transfer does not switch which of the two is better off. Gini egalitarianism satisfies Pigou–Dalton, so it ranks X above Y whatever person 3 has.

Row (iii). (a) X: 20. Y: $3 \cdot 3 + 1 \cdot 12 = 21$. So Y ranks higher. By the ratio: as in row (i), the gain has to be more than 6. It is 7.

(b) Take $z = 5$. Then $X = (5, 5, 5)$ gives $5 \cdot 5 + 3 \cdot 5 + 1 \cdot 5 = 25 + 15 + 5 = 45$, and $Y = (3, 5, 12)$ gives $5 \cdot 3 + 3 \cdot 5 + 1 \cdot 12 = 15 + 15 + 12 = 42$. So X ranks higher. By the ratio: with person 3 between them, persons 1 and 2 have the weights 5 and 1, a ratio of 5, so the gain would have to be more than $5 \times 2 = 10$. It is 7.

Question 7

Eleven people, A to K, have wellbeing 30,000, 31,000, ..., 40,000: A has 30,000, B has 31,000, and so on up to K with 40,000. Each option gives one of them a gift of wellbeing, and nobody else changes: "2 to B" means that B gets 2 more units of wellbeing.

Each version of prioritarianism in this question has a multiplier m, a number with $0 < m < 1$. It ranks options by how much they add to the priority-weighted sum. Each unit of wellbeing adds the following amount, depending on where it falls:

Stretch of wellbeing	Each unit adds
30,000 to 31,000	1
31,000 to 32,000	m
32,000 to 33,000	m^2
33,000 to 34,000	m^3

And so on: multiplied by m with each further 1,000 up. For example, with $m = 0.8$, the option "3 to C" adds $3 \times 0.8^2 = 1.92$.

You may use: 0.9^{10} is about 0.35, and $0.25^{10} = \dfrac{1}{1{,}048{,}576}$.

(a) Take $m = 0.9$.

(i) How does this version rank 1 to A and 2 to B?

(ii) How does it rank 1 to A and 1,000 to K?

Solution: Each gift stays within the stretch where its recipient starts. A's unit runs from 30,000 to 30,001, so 1 to A adds 1. B's two units run from 31,000 to 31,002, so 2 to B adds $2m$. K's 1,000 units run from 40,000 to 41,000, ten stretches above the first, where each unit adds m^{10}; so 1,000 to K adds $1{,}000\, m^{10}$.

(i) 2 to B adds $2 \times 0.9 = 1.8$, more than the 1 added by 1 to A. So 2 to B ranks above 1 to A.

(ii) 1,000 to K adds $1{,}000 \times 0.9^{10}$, about $1{,}000 \times 0.35 = 350$, more than 1. So 1,000 to K ranks above 1 to A.

(b) Take $m = 0.25$.

(i) How does this version rank 1 to A and 2 to B?

(ii) How does it rank 1 to A and 1,000 to K?

Solution: (i) 2 to B adds $2 \times 0.25 = 0.5$, less than 1. So 1 to A ranks above 2 to B.

(ii) 1,000 to K adds $1{,}000 \times 0.25^{10} = \dfrac{1{,}000}{1{,}048{,}576}$, less than 1. So 1 to A ranks above 1,000 to K.

(c) Is there a multiplier m for which the version ranks 1 to A above 2 to B, and also ranks 1,000 to K above 1 to A? If so, give one. If not, explain why not.

Solution: No. The version ranks 1 to A above 2 to B exactly when $1 > 2m$, that is, when $m < \frac{1}{2}$. But if $m < \frac{1}{2}$, then

$$m^{10} < \left(\tfrac{1}{2}\right)^{10} = \frac{1}{1{,}024},$$

so 1,000 to K adds less than $\dfrac{1{,}000}{1{,}024}$, which is less than 1. So it ranks 1 to A above 1,000 to K as well.