PE3101P: Decision and Social Choice

Optional Exercises 1 — Solutions

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1. For each of the following situations, write down whether the agent did the correct thing:

   (i) **only** in the fact-relative sense;

   (ii) **only** in the evidence-relative sense;

   (iii) in **both** senses; or

   (iv) in **neither** sense.

   (a) Basil receives a tip on the horse Dragonfly. His informant, who Basil knows to be very knowledgeable and reliable, tells Basil that Dragonfly is very likely to win its next race, despite the official odds being 14:1. Basil has exactly £5 to his name, and wagers all of it on Dragonfly. Dragonfly wins the race and Basil collects winnings of £70.

   Solution. **Both.** A very likely winner at 14:1 is a good bet on the evidence, and it won.

   (b) Petra is considering whether to spend $10,000 on a trip to Australia, or whether to instead save the money to cover her expenses after her current work contract expires (after which she anticipates she will need the money to maintain a reasonable standard of living, as she has no other savings). She decides that there's a small chance she could die at any time anyway, so she might as well enjoy the money while she can. She spends the money on the trip to Australia and has a fantastic time. Just before her work contract expires, by total coincidence, Petra dies in a freak accident.

   Solution. **Only in the fact-relative sense.** A small chance of dying at any moment does not justify spending money she expects to need. But she died before the contract expired, so the savings would have gone unused.

   (c) Tomi is considering whether to buy health insurance, which is offered at a heavily subsidised rate by his employer. He decides not to bother, as he would rather have more money to spend on noodles. The following year he falls seriously ill, and the resulting medical bills bankrupt him.

   Solution. **Neither.** Turning down heavily subsidised insurance against an outcome that would bankrupt him is a bad decision on the evidence, and he fell ill.

   (d) Zhuge Liang is commanding an army during a period of civil war, having been tasked to defend a city. He knows that his forces are no match for the oncoming enemy. The conventional move would be to retreat and minimise his own losses. Instead, he orders the few men he has to throw open the gates and disguise themselves as civilians as the enemy approach, wagering that the opposing commander, who is known to be very cautious, will falsely believe that he has set a massive ambush. The ruse works and the enemy retreats without fighting.

   Solution. **Both.** He knew the opposing commander was very cautious, and the alternative was to give up the city he had been ordered to defend. And it worked.

(e) Buffon is tired of waiting before crossing the road. He calculates that the probability of being hit if he simply runs across without looking is around $1/20{,}000$. Buffon decides that this is a pretty small chance of things going wrong, and decides to run across the road from now on in order to save time. He is hit by a truck and dies.

Solution. **Neither.** He died. And it is foolish to accept a $1/20{,}000$ chance of death to save a few seconds.

(f) The COVID pandemic is at its peak, and Eric has an immunodeficiency disorder which means he is at great risk of dying if he contracts COVID before he is able to take a vaccine. His friends ask him to come out in the evening to play cards. Although he would like to play, Eric plays it safe and refuses, citing his condition and noting that there is a good chance that at least one of them has the virus right now, perhaps asymptomatically. In fact, none of his friends have COVID at the moment.

Solution. **Only in the evidence-relative sense.** Given his condition and a real chance that someone was infectious, refusing was right on the evidence. In fact nobody had the virus, so going would have been fine.

(g) A politician is deciding how to prepare the country for various wartime scenarios. His planners present him with a cheap and highly effective way of protecting the majority of the country's citizens from the global effects of a large-scale nuclear war between third parties, which the planners reasonably assess to be fairly unlikely, but nevertheless a very significant risk this century. The politician elects not to implement the measure, instead using the money to provide a very small uplift in funding for the country's healthcare system. A nuclear war does not happen that century.

Solution. **Only in the fact-relative sense.** Giving up a cheap and highly effective measure against a significant risk, for a very small gain elsewhere, is a bad decision on the evidence. But no war occurred, so the measure would have bought nothing.

2. In each of the following, you are given the acts, the states of nature and the outcomes of a decision problem, jumbled together. In each case, work out what the decision problem must be, and sort the items into the three categories. If you like, you may also draw the decision tables, to check that you can — but that part is optional.

(a)
- The investor makes a very healthy return
- There is a stock market crash this year
- Buy bonds
- The investor is wiped out
- There is no stock market crash this year
- Buy stocks
- The investor makes a small return

Solution. **Where to put your money.**

Acts: buy stocks; buy bonds.
States: there is a stock market crash this year; there is no stock market crash this year.
Outcomes: the investor is wiped out; the investor makes a very healthy return; the investor makes a small return.

	Crash	**No crash**
Buy stocks	Wiped out	Very healthy return
Buy bonds	Small return	Small return

(b)
- You catch your flight, having spent \$30
- There is heavy traffic on the expressway
- Take a taxi
- You miss your flight, having spent \$30
- You catch your flight, having spent \$12
- There is no heavy traffic
- Take the airport train

Solution. **How to get to the airport.**

Acts: take a taxi; take the airport train.
States: there is heavy traffic on the expressway; there is no heavy traffic.
Outcomes: you miss your flight, having spent \$30; you catch your flight, having spent \$30; you catch your flight, having spent \$12.

	Heavy traffic	**No heavy traffic**
Take a taxi	Miss it, spent \$30	Catch it, spent \$30
Take the airport train	Catch it, spent \$12	Catch it, spent \$12

The train is unaffected by road traffic, so it gets you there in both states. You pay for the taxi whether or not it gets you there in time.

(c)
- A drought
- Plant a frost-resistant crop
- A normal harvest
- An early frost
- Plant a drought-resistant crop
- A normal season
- The crop dies

Solution. **Which crop to plant.**

Acts: plant a frost-resistant crop; plant a drought-resistant crop.
States: an early frost; a normal season; a drought.
Outcomes: a normal harvest; the crop dies.

	Early frost	**Normal season**	**Drought**
Frost-resistant crop	A normal harvest	A normal harvest	The crop dies
Drought-resistant crop	The crop dies	A normal harvest	A normal harvest

Each crop survives the condition it resists and a normal season, and dies in the other extreme.

3. Recall two conditions we placed on the states of nature in a decision table. First, together with the act chosen, a state must *completely* specify the resulting outcome, leaving nothing further up to chance. Second, the states must be independent of the acts: which act you choose cannot change which state of nature is actual, and cannot change how probable the states are.

In addition, the set of *all* the states of nature in a table must satisfy two further requirements:

– **Mutual exclusivity.** No two states can both obtain. If two could, then it would not be settled which cell of the table describes what actually happens.

– **Joint exhaustiveness.** Whatever happens, at least one state obtains. If some way the world might be has been left out, then there are possible results of your choice which the table simply fails to record.

Taken together, these two say that *exactly one* state of nature is always the actual one. Note that, unlike the two conditions in the previous paragraph, these are requirements on the states taken as a whole rather than on any individual state.

For each of the following decision tables, say what, if anything, is wrong with the states of nature it uses.

(a) Whether to bet on your own roll, when you and a friend each roll a six-sided die.

	Your number is greater	**Your friend's number is greater**
Bet \$5 on your roll	+\$5	−\$5
Don't bet	\$0	\$0

Solution. **Not jointly exhaustive.** The two dice can show the same number, and no column covers that.

(b) Whether to cycle to work or take the bus.

	It rains	**Traffic is heavy**	**Neither**
Cycle	Soaked, quick	Dry, quick	Dry, quick, pleasant
Take the bus	Dry, slow	Dry, very slow	Dry, slow

Solution. **Not mutually exclusive.** It can rain while the traffic is heavy, and then two columns apply at once and disagree. The states are exhaustive: "neither" covers the rest.

(c) Whether to travel across town by MRT or by Grab.

	There is a delay on the MRT line	**There is no delay**
Take the MRT	Arrive 30 minutes late; fare \$1.50	Arrive on time; fare \$1.50
Take a Grab	Arrive on time; fare \$18	Arrive on time; fare \$18

Solution. **Nothing is wrong with it.** There is a delay or there is not, never both and never neither; how you travel cannot affect whether the line is delayed; and each act together with each state fixes both when you arrive and what you pay.

(d) Whether to lock your bicycle when you leave it outside.

	The bicycle is stolen	**The bicycle is not stolen**
Lock it	Bicycle gone; a minute spent	Bicycle kept; a minute spent
Leave it unlocked	Bicycle gone; a minute saved	Bicycle kept; a minute saved

Solution. **The states are not independent of the acts.** Locking the bicycle affects whether it is stolen.

(e) Whether to buy a lottery ticket.

	The draw is fair	**The draw is rigged**
Buy a ticket	You win the jackpot, or you lose your stake	You lose your stake
Don't buy	Nothing happens	Nothing happens

Solution. **The states do not settle the outcome.** "Buy a ticket" together with "the draw is fair" leaves it open whether you win.

(f) Whether to train hard in the weeks before a race.

	Injured on race day	**Not injured**
Train hard	Cannot race	Best possible performance
Train lightly	Cannot race	Decent performance

Solution. **The states are not independent of the acts.** Training hard does not determine whether you are injured, but it raises the probability, which is enough.

4. (a) Recall that, for any two acts A and B:

- A **strongly statewise dominates** B if A results in a better outcome than B in *every* state of nature.
- A **weakly statewise dominates** B if A results in an outcome at least as good as B's in *every* state of nature, and a better outcome in *at least one* state of nature.

Explain why, if A strongly statewise dominates B, it follows that A also weakly statewise dominates B.

Solution. Suppose A strongly statewise dominates B.

In every state, A's outcome is better than B's.	[definition of strong dominance]
So in every state, A's outcome is at least as good as B's.	[better implies at least as good]
And in at least one state, A's outcome is better than B's.	[there is at least one state]
So A weakly statewise dominates B.	[definition of weak dominance]

(b) In each of the tables below, the numbers in the cells are **utilities**: the higher the number, the better that outcome is by the agent's lights. The states of nature are written S_1, S_2, and so on, and the probability of each state is given at the head of its column.

Identify every instance of weak or strong statewise dominance. In light of part (a), when one act strongly statewise dominates another you need not also say that it weakly statewise dominates it.

(i)

	S_1 $(p = 0.6)$	S_2 $(p = 0.4)$
A_1	10	4
A_2	6	2
A_3	6	4

Solution. A_1 strongly dominates A_2. A_1 weakly dominates A_3 (better in S_1, equal in S_2). A_3 weakly dominates A_2 (equal in S_1, better in S_2). A_2 dominates nothing.

(ii)

	S_1 $(p = 0.90)$	S_2 $(p = 0.07)$	S_3 $(p = 0.03)$
A_1	100	100	0
A_2	90	90	5
A_3	95	80	2

Solution. **Nothing dominates anything.** A_1 is better than both A_2 and A_3 in S_1 and S_2 but worse in S_3. A_2 is better than A_3 in S_2 and S_3 but worse in S_1. Dominance takes no account of how probable the states are.

(iii)

	S_1 $(p = 0.25)$	S_2 $(p = 0.75)$
A_1	8	8
A_2	8	8
A_3	8	6
A_4	9	9

Solution. A_4 strongly dominates A_1, A_2 and A_3. A_1 weakly dominates A_3, and so does A_2. Neither of A_1 and A_2 dominates the other: their payoffs are identical, and weak dominance requires being better in at least one state.

5. Recall that:

- An **event** is a set of possible outcomes.
- X is the set of all possible outcomes: the event that *something* happens.
- $\mathrm{Cr}(\sigma)$ denotes an agent's credence in the event σ.

The probability axioms say that, for any events σ and τ:

- **Non-negativity:** $\mathrm{Cr}(\sigma) \geq 0$.
- **Normalisation:** $\mathrm{Cr}(X) = 1$.
- **Additivity:** if σ and τ are **disjoint**, then $\mathrm{Cr}(\sigma \cup \tau) = \mathrm{Cr}(\sigma) + \mathrm{Cr}(\tau)$.

Throughout, $\neg\sigma$ is shorthand for the *complement* of σ: the event that no outcome in σ occurs. Note that, for any event σ, the events σ and $\neg\sigma$ are disjoint, and $\sigma \cup \neg\sigma = X$. You will not be required to show this, and may assume it freely.

Worked example. *Show that* $\mathrm{Cr}(\sigma) + \mathrm{Cr}(\neg\sigma) = 1$.

$$
\begin{aligned}
\mathrm{Cr}(X) &= 1 && \text{[Normalisation]} \\
\mathrm{Cr}(X) &= \mathrm{Cr}(\sigma \cup \neg\sigma) && \text{[substitution of identicals, since } X = \sigma \cup \neg\sigma\text{]} \\
\mathrm{Cr}(\sigma \cup \neg\sigma) &= \mathrm{Cr}(\sigma) + \mathrm{Cr}(\neg\sigma) && \text{[Additivity, since } \sigma \text{ and } \neg\sigma \text{ are disjoint]} \\
\mathrm{Cr}(\sigma) + \mathrm{Cr}(\neg\sigma) &= 1 && \text{[from the three lines above]}
\end{aligned}
$$

which is what we wanted. Rearranging gives a form that is often more convenient: $\mathrm{Cr}(\neg\sigma) = 1 - \mathrm{Cr}(\sigma)$.

Notice what the proof does. Every line is either an axiom, or a fact about the events themselves, or something that follows from lines already written — and each line says which. Prove the following in the same style. You may use the worked example above, and any result you have already proved, in later proofs.

(a) Show that $\mathrm{Cr}(\varnothing) = 0$, where $\varnothing$ is the empty event: the event containing no possible outcomes at all.

Solution. $\varnothing$ and X are complements, so:

$$\begin{aligned}
\mathrm{Cr}(\varnothing) &= \mathrm{Cr}(\neg X) && [\varnothing = \neg X] \\
&= 1 - \mathrm{Cr}(X) && [\text{worked example}] \\
&= 1 - 1 = 0 && [\text{Normalisation}]
\end{aligned}$$

(b) Show that $\mathrm{Cr}(\sigma) \leq 1$, for every event σ.

Hint: you may find it easiest to argue by contradiction. That means assuming that what you are trying to prove is false, and deriving a contradiction from that assumption; if you can do so, you have thereby shown that what you were trying to prove is true.

Solution. Suppose for contradiction that $\mathrm{Cr}(\sigma) > 1$.

$$\begin{aligned}
\mathrm{Cr}(\neg\sigma) &= 1 - \mathrm{Cr}(\sigma) && [\text{worked example}] \\
&< 0 && [\text{since } \mathrm{Cr}(\sigma) > 1]
\end{aligned}$$

But $\mathrm{Cr}(\neg\sigma) \geq 0$ by Non-negativity. Contradiction; so $\mathrm{Cr}(\sigma) \leq 1$.

(c) Suppose σ, τ and ρ are pairwise disjoint — that is, no two of them have any outcome in common. Show that

$$\mathrm{Cr}(\sigma \cup \tau \cup \rho) = \mathrm{Cr}(\sigma) + \mathrm{Cr}(\tau) + \mathrm{Cr}(\rho).$$

Hint: you may assume that if σ and τ are each disjoint from ρ, then $\sigma \cup \tau$ is disjoint from ρ.

Solution. σ and τ are each disjoint from ρ, so $\sigma \cup \tau$ is disjoint from ρ. Hence:

$$\begin{aligned}
\mathrm{Cr}(\sigma \cup \tau \cup \rho) &= \mathrm{Cr}\left((\sigma \cup \tau) \cup \rho\right) && [\text{associativity of union}] \\
&= \mathrm{Cr}(\sigma \cup \tau) + \mathrm{Cr}(\rho) && [\text{Additivity, } \sigma \cup \tau \text{ and } \rho \text{ disjoint}] \\
&= \mathrm{Cr}(\sigma) + \mathrm{Cr}(\tau) + \mathrm{Cr}(\rho) && [\text{Additivity, } \sigma \text{ and } \tau \text{ disjoint}]
\end{aligned}$$

Bracketing the other way, as $\sigma \cup (\tau \cup \rho)$, would do just as well.

(d) Let $\sigma \subseteq \tau$ mean that σ is a *subset* of τ: every outcome in σ is also in τ. Let $\tau \setminus \sigma$ be the set of outcomes that are in τ but not in σ.

Show that if $\sigma \subseteq \tau$, then $\mathrm{Cr}(\sigma) \leq \mathrm{Cr}(\tau)$.

Hint: if $\sigma \subseteq \tau$ then $\tau = \sigma \cup (\tau \setminus \sigma)$, and these two events are disjoint. You are not required to show this, and may assume it.

Solution.

$$\begin{aligned}
\mathrm{Cr}(\tau) &= \mathrm{Cr}\left(\sigma \cup (\tau \setminus \sigma)\right) && [\text{given, since } \sigma \subseteq \tau] \\
&= \mathrm{Cr}(\sigma) + \mathrm{Cr}(\tau \setminus \sigma) && [\text{Additivity, the two being disjoint}] \\
&\geq \mathrm{Cr}(\sigma) && [\text{Non-negativity}]
\end{aligned}$$

(e) Let $\sigma \cap \tau$ be the set of outcomes that are in both σ and τ. Show that

$$\mathrm{Cr}(\sigma) = \mathrm{Cr}(\sigma \cap \tau) + \mathrm{Cr}(\sigma \cap \neg\tau).$$

Hint: $\sigma = (\sigma \cap \tau) \cup (\sigma \cap \neg\tau)$, *and these two events are disjoint. Again, you may assume this.*

Solution.

$$\begin{aligned} \mathrm{Cr}(\sigma) &= \mathrm{Cr}\left((\sigma \cap \tau) \cup (\sigma \cap \neg\tau)\right) && \text{[given]} \\ &= \mathrm{Cr}(\sigma \cap \tau) + \mathrm{Cr}(\sigma \cap \neg\tau) && \text{[Additivity, the two being disjoint]} \end{aligned}$$

(f) Show that

$$\mathrm{Cr}(\sigma \cup \tau) = \mathrm{Cr}(\sigma) + \mathrm{Cr}(\tau) - \mathrm{Cr}(\sigma \cap \tau).$$

Hint: you may assume both of the following. First, $\sigma \cup \tau = \sigma \cup (\tau \setminus \sigma)$, *and these two events are disjoint. Second,* $\tau = (\tau \setminus \sigma) \cup (\sigma \cap \tau)$, *and these two events are disjoint as well.*

Solution. First:

$$\begin{aligned} \mathrm{Cr}(\sigma \cup \tau) &= \mathrm{Cr}\left(\sigma \cup (\tau \setminus \sigma)\right) && \text{[given]} \\ &= \mathrm{Cr}(\sigma) + \mathrm{Cr}(\tau \setminus \sigma) && \text{[Additivity, the two being disjoint]} \end{aligned}$$

Next:

$$\begin{aligned} \mathrm{Cr}(\tau) &= \mathrm{Cr}\left((\tau \setminus \sigma) \cup (\sigma \cap \tau)\right) && \text{[given]} \\ &= \mathrm{Cr}(\tau \setminus \sigma) + \mathrm{Cr}(\sigma \cap \tau) && \text{[Additivity, the two being disjoint]} \end{aligned}$$

so $\mathrm{Cr}(\tau \setminus \sigma) = \mathrm{Cr}(\tau) - \mathrm{Cr}(\sigma \cap \tau)$. Substituting into the first result:

$$\mathrm{Cr}(\sigma \cup \tau) = \mathrm{Cr}(\sigma) + \mathrm{Cr}(\tau) - \mathrm{Cr}(\sigma \cap \tau).$$