Decision and Social Choice

PE3101P — Lecture 10: The Liberal Paradox

Instructor: Tomi Francis

National University of Singapore

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Today

1. Choice Functions
2. The Liberal Paradox
3. When Is It a Paradox?
4. Rights as Choices

From Wellbeing to Preferences

- The last three weeks: welfarist social choice. We often compared how well off different people are.
- Now we mostly drop that information, or focus less on it.
- Instead: the social aggregation of people's *preferences*.
- Voting theory works this way too.

Social States and Preferences

- A social state: a complete description of how things are for everyone.
- Each person i has a preference relation $\succsim_i$ over social states: an ordinary preference relation, as in the decision theory half of the course.
- Preferences can be about other people's business. You can prefer one social state to another because of what somebody else does in it.

Menus and Choice Functions

- A menu S: a set of available social states.
- $C(S)$: what society may choose from S. It is a non-empty subset of S.
- A choice function C gives a $C(S)$ for every menu S.
- For now, we leave open what "may choose" means. The third section of today's lecture looks at different readings of it.

From a Ranking to a Choice Function

- A ranking $\succsim$ of social states gives a choice function: from any menu, choose the *maximal* options, the ones that nothing on the menu is strictly better than.
- In symbols:

$$C(S) = \{x \in S : \text{there is no } y \in S \text{ with } y \succ x\}.$$

- If $\succ$ has no cycles (Acyclicity), every finite menu has at least one maximal option. Transitivity isn't needed.
- An *optimal* option is at least as good as everything on the menu. If $\succsim$ is complete, the maximal options and the optimal options are the same.

From a Choice Function to a Ranking

- From any choice function C, we can read off a ranking: x is at least as good as y when x is chosen from $\{x, y\}$. This ranking is complete, since something is always chosen from $\{x, y\}$.
- *C comes from a ranking* when choosing the maximal options by this ranking gives back C, from every menu. No other complete ranking could do that: one that ranked some pair differently would choose differently from that pair.
- Not every choice function comes from a ranking.
- So using C doesn't assume that what else is on the menu is irrelevant to the choice between two options.

Contraction Consistency (Property α)

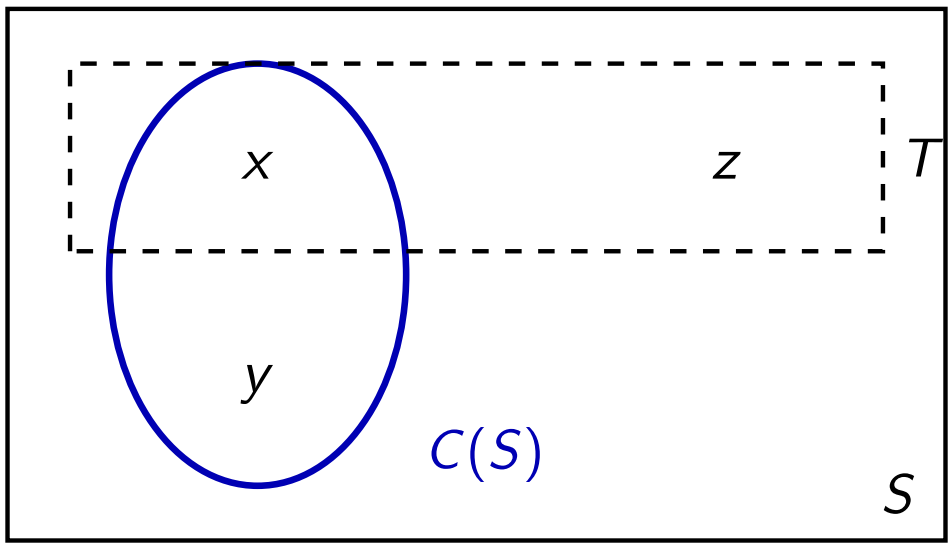
Contraction consistency (Property α)

If x is chosen from a menu S, and x is on a smaller menu T inside S, then x is chosen from T.

$$[x \in C(S) \text{ and } x \in T \subset S] \Rightarrow x \in C(T)$$

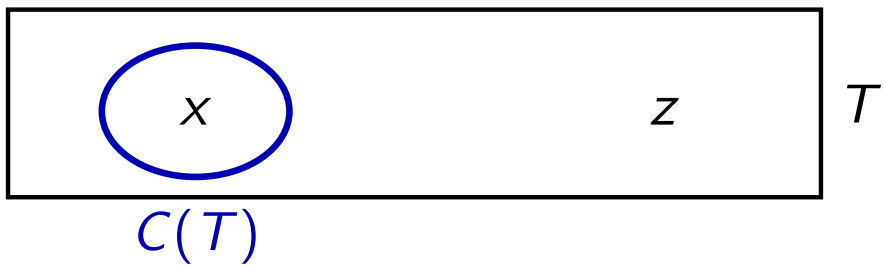
- A choice function that comes from a ranking satisfies contraction consistency. If nothing on S is strictly better than x, nothing on the smaller menu T is either.

Contraction Consistency, Pictured



Now shrink the menu to $T = \{x, z\}$:
y is no longer on it.

Contraction Consistency, Pictured



x was chosen from S, and x is still on T.
Contraction consistency: x is still chosen from T.
(It says nothing about z.)

Contraction Consistency: An Example

- Three slices of cake: z is the largest, then y, then x.[1]

Menu	He chooses

He takes as large a slice as possible, but never the largest: "never pick the largest slice".

[1] Sen (1993: 501).

Contraction Consistency: An Example

- Three slices of cake: z is the largest, then y, then x.[1]

Menu	He chooses
$\{x, y\}$	x, the smaller slice
$\{x, y, z\}$	y, the middle slice

y is chosen from $\{x, y, z\}$, and y is on the smaller menu $\{x, y\}$, but it isn't chosen there: contraction consistency fails.

[1] Sen (1993: 501).

Expansion Consistency (Property γ)

Expansion consistency (Property γ)

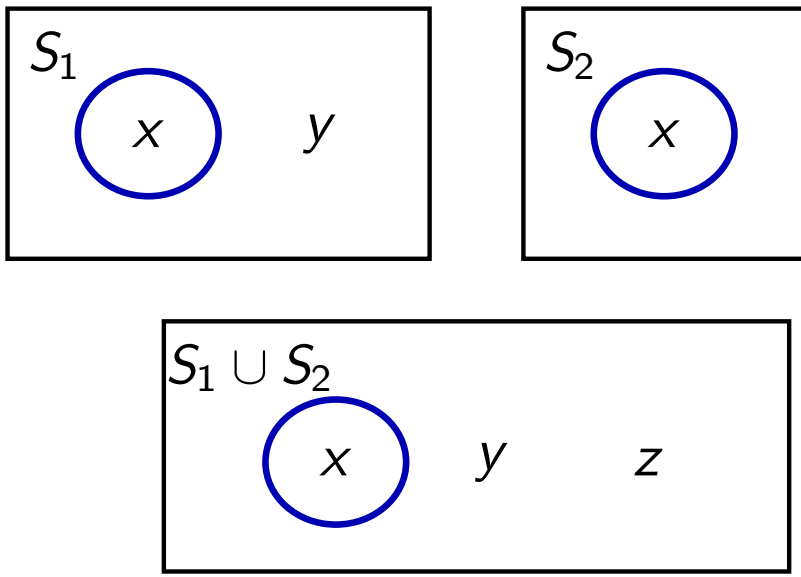
If x is chosen from a menu S_1, and x is chosen from a menu S_2, then x is chosen from the menu that combines them.

$$[x \in C(S_1) \text{ and } x \in C(S_2)] \Rightarrow x \in C(S_1 \cup S_2)$$

- For finitely many menus, this is the same as Sen's version for any number of menus:[1] apply it again and again.

[1] Sen (1971: 314).

Expansion Consistency, Pictured



Expansion consistency: x may be chosen from $S_1 \cup S_2$ too.

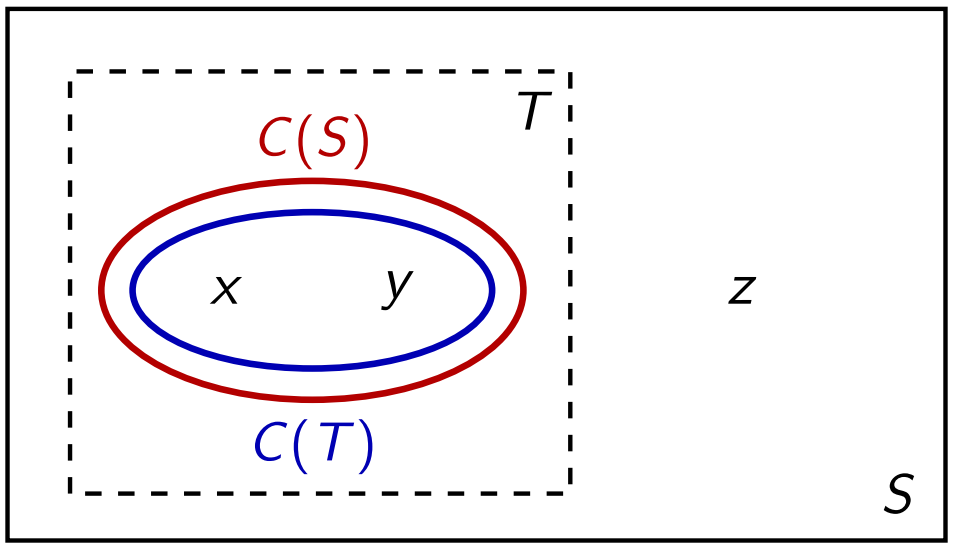
Property β

Property β

If x and y are both chosen from T, and $T \subset S$, then y is chosen from S if and only if x is.

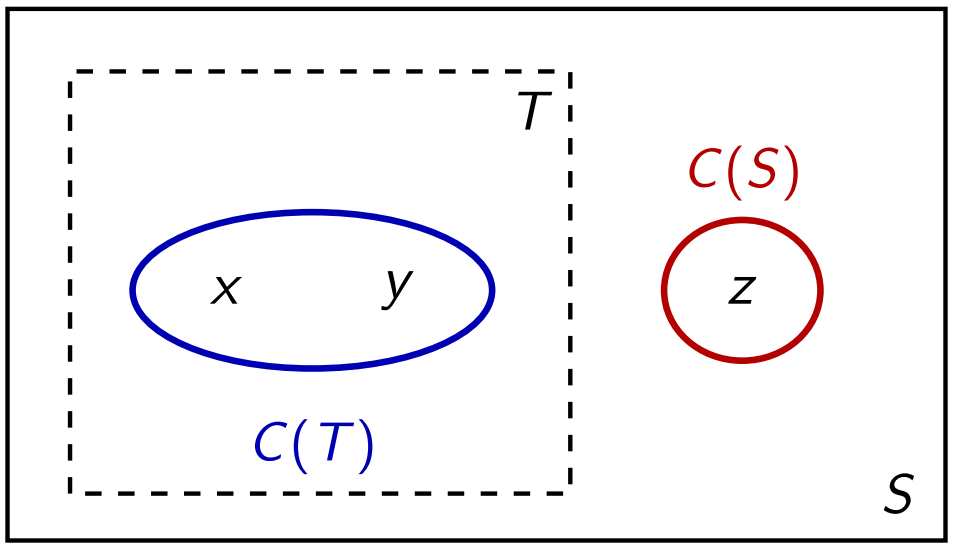
$$[x, y \in C(T) \text{ and } T \subset S] \Rightarrow [y \in C(S) \Leftrightarrow x \in C(S)]$$

Property β, Pictured



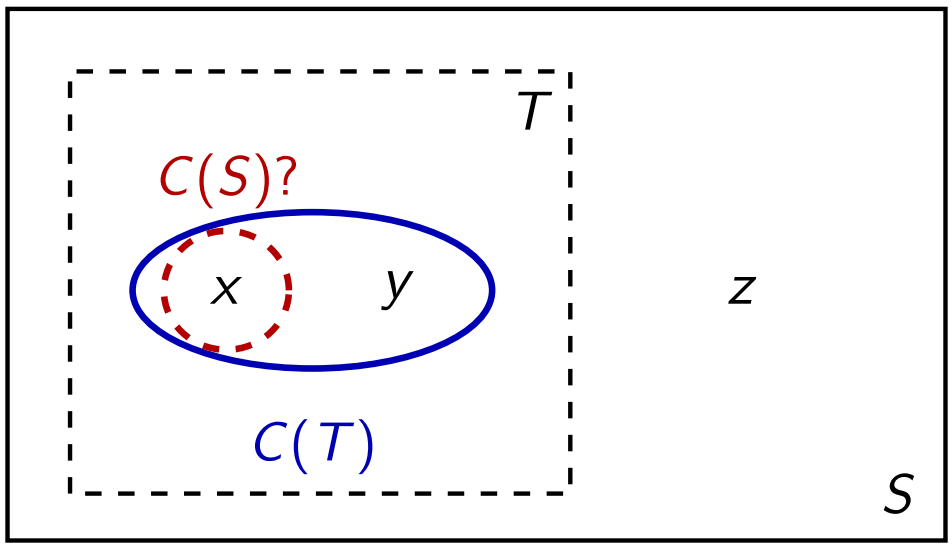
Allowed: x and y both chosen from S.

Property β, Pictured



Also allowed: neither of them chosen from S.

Property β, Pictured



Ruled out by Property β: one of them chosen from S without the other.
From S, it's both or neither.

What the Properties Show

- Sen showed, for C defined on every non-empty finite menu:
  - contraction and expansion consistency (α and γ) hold exactly when C comes from a ranking;[1]
  - contraction consistency and Property β (α and β) hold exactly when C comes from a transitive ranking.[2]

[1] Sen (1971: 314).
[2] Sen (1971: 310, 314).

Pareto for Choice Functions

- The liberal paradox uses two conditions on a choice function: Pareto, and a condition of liberty.
- This version of Pareto is about preferences, not wellbeing. And it is written with the choice function C.

Pareto

If everyone strictly prefers x to y, then y is not chosen from any menu containing x.

$$x \succ_i y \text{ for every } i, \text{ and } x, y \in S \;\Rightarrow\; y \notin C(S)$$

Liberty in Personal Matters

- Some choices seem to be nobody's business but your own. For example:[1]
- If you prefer your walls pink rather than white, society should let you have them pink, even if a majority would like to see them white.
- Whether you sleep on your back or on your belly should be up to you, even if a majority would rather you slept on your back.
- Sen states this kind of liberty in a very weak form.[1]

[1] Sen (1970: 152).

Minimal Liberalism

Decisiveness

A person is *decisive* over a pair of social states $\{x, y\}$ when, whichever of the two they prefer, the other is not chosen from any menu containing both.

If $x \succ_i y$, then $y \notin C(S)$ for every menu S containing x and y; and if $y \succ_i x$, then $x \notin C(S)$ for every such S.

Minimal Liberalism

There are at least two people who are each decisive over at least one pair of social states.

What Minimal Liberalism Asks

- Informally, the two states in a person's pair differ only in a personal matter: the colour of that person's walls, say. Formally, the pair can be any pair.
- The person wins in either direction: whichever of the two they prefer.
- "Minimal": only two people, and only one pair each.

The Book Case

- A case with Pareto and Minimal Liberalism together.[1] One copy of *Lady Chatterley's Lover*; two people, Prude and Lewd.[2]

State	What happens
x	Prude reads it
y	Lewd reads it
z	Nobody reads it

- Prude would most like nobody to read it. But rather than Lewd reading it, he would read it himself.
- Lewd would most like Prude to read it; next, to read it himself; last, nobody.

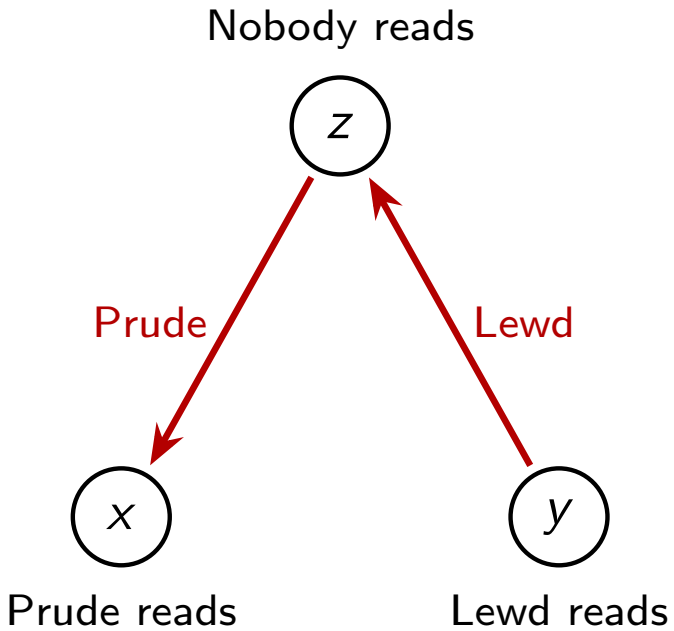
[1] Sen (1970: 155).
[2] Sen (2017: 434).

Prude's and Lewd's Preferences and Rights

	Prude	Lewd
Best	Nobody reads (z)	Prude reads (x)
Middle	Prude reads (x)	Lewd reads (y)
Worst	Lewd reads (y)	Nobody reads (z)

- Prude's right: he is decisive over $\{x, z\}$, whether he reads it.
- Lewd's right: he is decisive over $\{y, z\}$, whether he reads it.

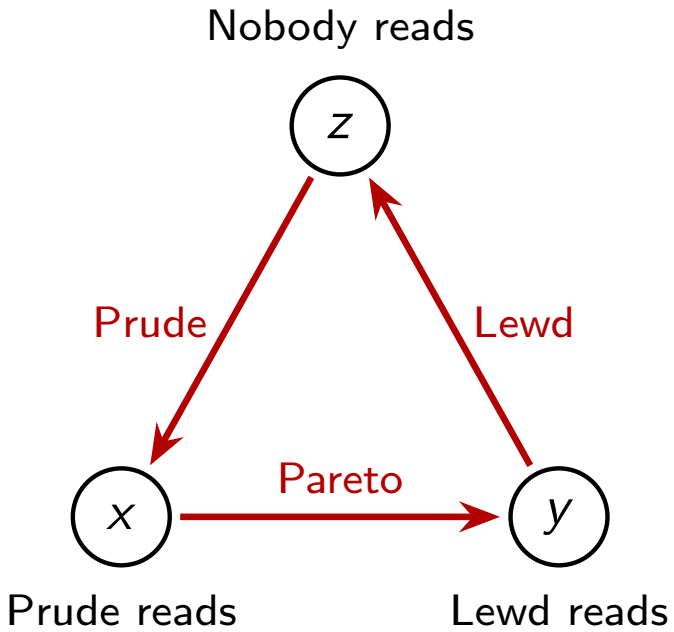
The Book Case, Menu by Menu



Menu	Excluded	May be chosen
$\{x, z\}$	Prude reads (x), by Prude's right	Nobody reads (z)
$\{y, z\}$	Nobody reads (z), by Lewd's right	Lewd reads (y)

Menu $\{y, z\}$: Lewd prefers reading it to nobody reading it.
His right excludes Nobody reads (z).
So only Lewd reads (y) may be chosen.

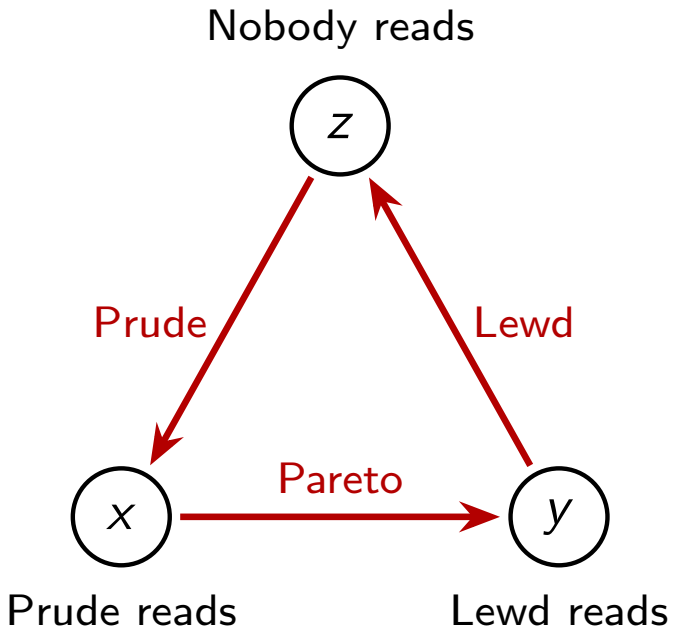
What May Be Chosen From $\{x, y\}$?



Menu	Excluded	May be chosen
$\{x, z\}$	Prude reads (x), by Prude's right	Nobody reads (z)
$\{y, z\}$	Nobody reads (z), by Lewd's right	Lewd reads (y)
$\{x, y\}$	Lewd reads (y), by Pareto	Prude reads (x)

Prude and Lewd both prefer Prude reads (x) to Lewd reads (y).
Pareto excludes Lewd reads (y).
So only Prude reads (x) may be chosen.

The Book Case: The Whole Menu



Menu	Excluded	May be chosen
$\{x, z\}$	Prude reads (x), by Prude's right	Nobody reads (z)
$\{y, z\}$	Nobody reads (z), by Lewd's right	Lewd reads (y)
$\{x, y\}$	Lewd reads (y), by Pareto	Prude reads (x)
$\{x, y, z\}$	every state, by all three	nothing

Prude's right excludes x, Pareto excludes y, Lewd's right excludes z.
Every state is excluded.
So nothing can be chosen.

Sen's Result

The liberal paradox

If any preferences at all are allowed (unrestricted domain), no way of choosing satisfies both Pareto and Minimal Liberalism.

- A way of choosing: a rule that gives a choice function, with something chosen from every menu, for every combination of people's preferences.
- The book case is one instance. Sen proves the result in three cases.[1]

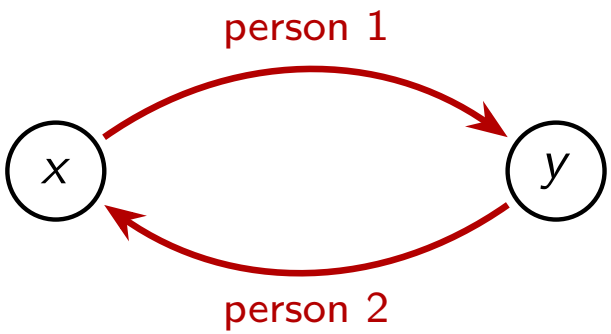
[1] Sen (1970: 154).

Sen's Proof: Three Cases

- Minimal Liberalism gives two people: person 1, decisive over one pair, and person 2, decisive over another.
- The two pairs are the same, or they share one option, or they share none.
- In each case, we can pick preferences, since any preferences are allowed, so that every option on some menu is excluded.[1]

[1] Sen (1970: 154).

Case 1: The Same Pair



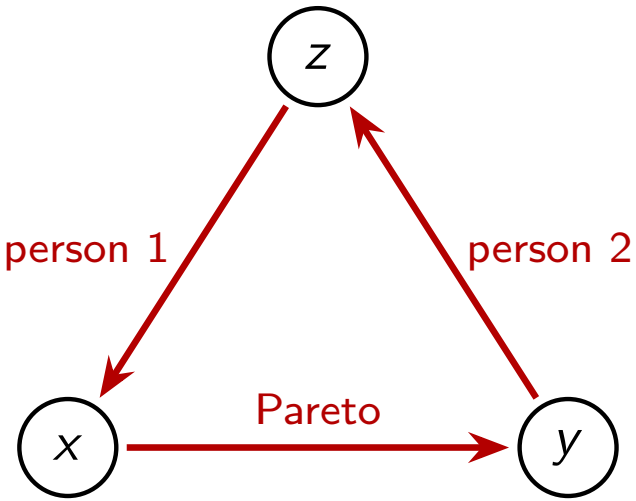
Both people decisive over $\{x, y\}$.

Person 1: $x \succ y$
Person 2: $y \succ x$[1]

Nothing can be chosen from $\{x, y\}$.
This case doesn't need Pareto.

[1] Sen (1970: 154).

Case 2: One Shared Option



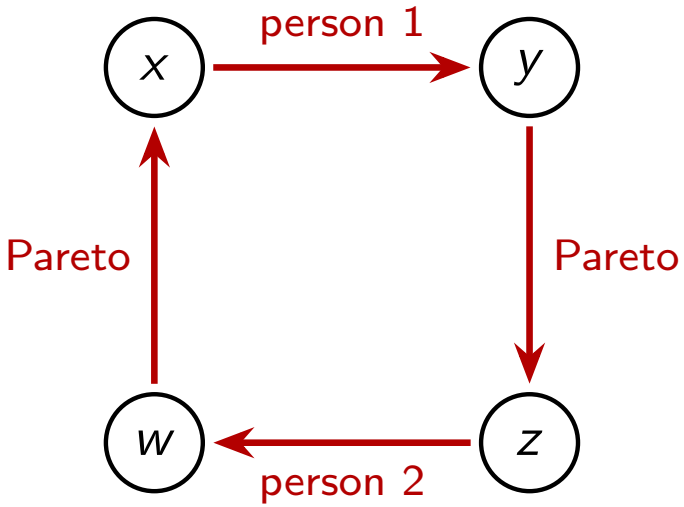
Person 1 decisive over $\{x, z\}$.
Person 2 decisive over $\{y, z\}$.

Person 1: $z \succ x \succ y$
Person 2: $x \succ y \succ z$
Everyone else: x above y.[1]

Nothing can be chosen from $\{x, y, z\}$.
These are Prude's and Lewd's preferences: the book case.

[1] Sen (1970: 154).

Case 3: Four Different Options



Person 1 decisive over $\{x, y\}$.
Person 2 decisive over $\{z, w\}$.

Person 1: $w \succ x \succ y \succ z$
Person 2: $y \succ z \succ w \succ x$
Everyone else: w above x,
y above z.[1]

Every option is excluded.
Nothing can be chosen from $\{x, y, z, w\}$.

[1] Sen (1970: 154).

What the Proof Uses

- In every case, each option on the menu is excluded, by someone's right or by Pareto.
- The result doesn't assume that choices come from a ranking, or contraction consistency (Property α), or transitivity: only that something can be chosen from every menu.

Four Readings of "Not Chosen"

In the first section, we left open what "may be chosen" means. "y is not chosen when x is available" can be read in four ways:

1. *Outcome evaluation*:[1] x is a better state of affairs than y.
2. *Deontic*: y ought not to be chosen when x is available.
3. *Normative choice*:[1] decision-making in society ought to be organised so that y is not chosen when x is available. A two-level claim, about how decision-making is set up.
4. *Descriptive choice*:[2] decision-making is organised so that y will not be chosen when x is available.

- In Lecture 8's terms: (1) is the telic reading; (2) is the deontic one.
- Sen: (3) and (4) don't require choices to come from a ranking.[2]

[1] Sen (1983: 6).
[2] Sen (1983: 7).

The Betterness Reading

- On reading (1), the whole theorem is not much of a paradox: Minimal Liberalism isn't plausible as a claim about betterness.
- Principles of liberty: best thought of as claims about rights.
- There may be reasons not to count a busybody's preferences in deciding what happens.
- But satisfying them seems to make things better: say, costlessly giving Lewd the thought that Prude is reading it. Denying that takes a messy, untidy consequentialism.
- Holding Minimal Liberalism about betterness means being a consequentialist about rights: not a popular view. Rights are typically deontic, not axiological.

The Deontic Reading

- On reading (2), each condition of the theorem is hard to give up.
- Unrestricted domain is hard to deny: you might simply be in that situation.
- Pareto is very plausible: everyone agrees. And it's supposed to express liberalism, or welfarism.
- Respecting people's liberty in some way seems right.
- So on this reading, it looks like a genuine paradox.

The Other Two Readings

- *Normative choice*: no good way of organising decision-making that always chooses something, whatever people's preferences, satisfies both Pareto and Minimal Liberalism.[1] Here Minimal Liberalism is weaker than giving people control over the choice.[2]
- *Descriptive choice*: no actual way of making decisions that always chooses something satisfies both.[1] Leave Prude and Lewd free to decide, and Lewd reads it: a state both rank below Prude reading it.[3]

[1] Sen (1983: 10).
[2] Sen (1983: 9).
[3] Sen (2017: 435).

Ways Out

- Restrict which preferences count: for example, only "tolerant" or "non-meddlesome" preferences.[1]
- Weaken Pareto.
  - Sen: a unanimous preference should decide only if everyone wants their preference to count; a person who respects others' rights won't want their meddlesome preferences to count.[2]
  - If even one person respects the rights of others, this weaker Pareto principle is consistent with liberty, though some ordinary Pareto preferences are overridden.[3]
- Reformulate liberty. The next section looks at one way of doing this.

[1] Sen (1983: 12).
[2] Sen (1976: 236).
[3] Sen (1976: 236–237).

Rights as Automatically Exercised

- Minimal Liberalism treats rights as automatically exercised: a person's preference over their pair is simply imposed on society.
- Gibbard does the same with features. A social state is a list of features, one per person: the colour of my walls, the colour of yours. Of two states that differ only in your feature, the one you prefer excludes the other.[1]
- But a right is something you may exercise or not, and perhaps strategically: players in a game don't always do what is immediately better for them.

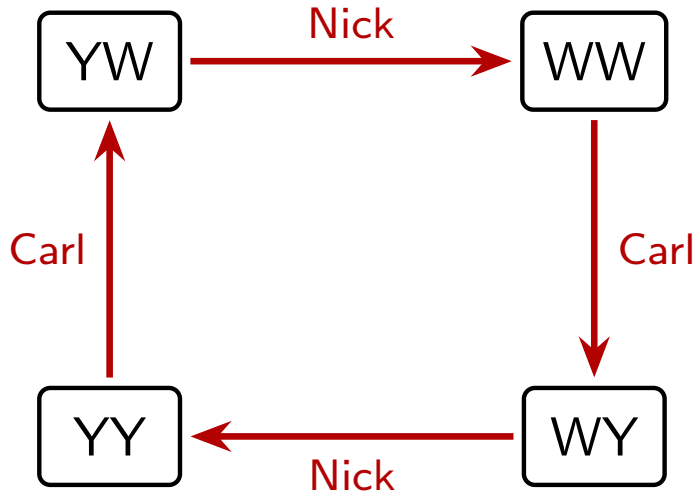
[1] Gibbard (1974: 389–391).

Exercising a Right, Not Waiving It

- You have the right to choose the colour of your T-shirt.
- If you choose the blue T-shirt, you have exercised your right to choose your T-shirt's colour. You haven't waived a right to the green one.
- Gibbard's own word is "waive".[1] But the cases he discusses aren't really waivers: they're about whether, and how, to exercise a right.

[1] Gibbard (1974: 399–400).

Gibbard's Paradox: The Walls



First letter: Nick's walls. Second: Carl's. W: white; Y: yellow.

Nick the nonconformist wants his walls a different colour from Carl's.

Carl the conformist wants his the same colour as Nick's.[1]

Every state is excluded, so nothing can be chosen.
Every arrow comes from a right: no Pareto.
The trouble is in the rights condition itself.

[1] Gibbard (1974: 389).

Gibbard's Ways Out

- Gibbard lets a right lapse when using it would only lead, through other people's rights and Pareto, to something the person likes no better.[1] He shows that rights over your own features that work like this (his "alienable" rights) are consistent with Pareto, as long as there are at least as many features as people.[2]
- Another: count only *unconditional* preferences, such as preferring white walls whatever colour the other person's walls are. That removes the walls paradox, but Sen's paradox comes back.[3]

[1] Gibbard (1974: 399–400).
[2] Gibbard (1974: 401).
[3] Gibbard (1974: 393–395).

Sugden's Objection

- Sugden takes the idea of rights as choices further: he objects to how Sen states liberty.[1] Sen's condition ties social choice to people's preferences over whole social states: to what each person would choose between two social states.
- Usually, people don't choose between social states. They choose features of them, such as the colour of their own walls. So these choices are only hypothetical.
- On the *control* view of liberty, liberty is about what people are actually left free to do.
- Sugden: "It is his choice that must be respected, not his preference".[2]

[1] Sugden (1985: 214–218).
[2] Sugden (1985: 217).

Game Forms

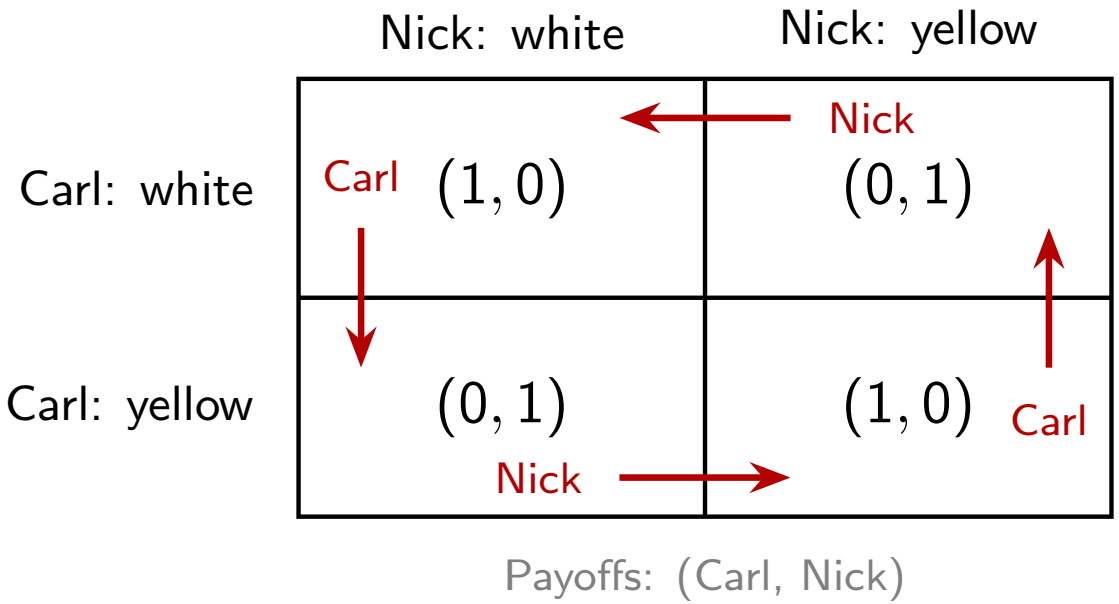
Game form

The players; each player's strategies; and an outcome for each combination of strategies. No preferences.[1]

- A right: a strategy that guarantees some feature of the outcome, whatever the others do.[2]
- So rights are a property of the game form, whatever people's preferences.

[1] Sugden (1985: 219).
[2] Sugden (1985: 225–226).

The Walls as a Game Form



But as a game it still has an outcome: each player just picks a colour.
Both have their rights, whatever they prefer.

Annie, Bill and Charlie

- Three states: x, nobody marries; y, Annie marries Bill, and Charlie stays single; z, Annie marries Charlie, and Bill stays single.[1]

	Annie	Bill	Charlie
Best	y	x	z
Middle	z	y	y
Worst	x	z	x

- Everyone has the right to stay single. On Sen's way of stating rights, Bill's preferring x to y is enough for x over y.[2]
- So y should not be chosen when x is available.

[1] Sugden (1985: 220).
[2] Sugden (1985: 220–221).

Bill Exercises His Right

- The marriage game: the men propose, or not; Annie accepts or refuses.
- Charlie's best choice is to propose. Annie's is to accept Bill or, failing him, Charlie.
- So if Bill proposes, Annie marries him (y); if not, she marries Charlie (z). Bill's best choice is to propose, and the outcome is y.[1]
- On Sen's way of stating rights, this fails to respect Bill's liberty: x was available, Bill prefers it, and y is chosen.
- But nobody invaded Bill's personal sphere. Bill chose to marry Annie: he exercised his right.[2]

[1] Sugden (1985: 221).
[2] Sugden (1985: 222).

Prude and Lewd as a Game

- Prude and Lewd both have rights over whether they themselves read the book.

	Lewd reads	Lewd doesn't
Prude reads	Both read	Prude only
Prude doesn't	Lewd only	Nobody reads

(i) All four combinations are available. The cooperative solution: Prude reads, and Lewd doesn't.

(ii) The only options are "Prude reads" and "nobody reads". A degenerate game: Prude is the only effective player. Prude doesn't read.

What the Game Solution Breaks

- Prude reads is chosen from the bigger menu in (i), and it is on the smaller menu in (ii), but it is not chosen there. So contraction consistency (Property α), from the first section, fails.
- Sen's rights condition fails too. In (i), Prude reads is chosen although Nobody reads was available, and Prude prefers Nobody reads.

Liberty Can Still Be Inefficient

- Game forms remove the paradox. But procedures that respect rights can still produce Prisoner's Dilemma outcomes: outcomes that everyone ranks below another available outcome.[1]
- For example: taking part in politics, by joining a party, going to meetings, perhaps voting itself.[2] We all want democratic politics to continue, but would rather other people did the work.
- Perhaps we would all be better off if we all took more part. But a law compelling everyone to take part would invade their liberty.
- As on the descriptive reading in the third section: leaving people free can lead to a state that everyone ranks below another.

[1] Sugden (1985: 227).
[2] Sugden (1985: 228).

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