Decision and Social Choice

PE3101P — Lecture 11: Arrow's Theorem

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Today

1. Setup
2. The Extreme Option Lemma
3. Proof of Arrow's Theorem
4. Voting Systems
5. Gibbard–Satterthwaite

Majority Voting

- Two options, x and y. Society ranks x above y when more voters prefer x to y.
- With two options, majority voting looks ideal.
- With three or more options, compare each pair by majority.

Condorcet's Paradox

Condorcet's profile[1]

Voters	Ranking
Voter 1	$x \succ y \succ z$
Voter 2	$y \succ z \succ x$
Voter 3	$z \succ x \succ y$

Three voters, three options.
How does majority voting rank each pair?

[1] Condorcet (1785).

Condorcet's Paradox

Condorcet's profile[1]

Voters	Ranking
Voter 1	$x \succ y \succ z$
Voter 2	$y \succ z \succ x$
Voter 3	$z \succ x \succ y$

x against y: voters 1 and 3 prefer x.
So $x \succ y$.

[1] Condorcet (1785).

Condorcet's Paradox

Condorcet's profile[1]

Voters	Ranking
Voter 1	$x \succ y \succ z$
Voter 2	$y \succ z \succ x$
Voter 3	$z \succ x \succ y$

y against z: voters 1 and 2 prefer y.
So $y \succ z$.

[1] Condorcet (1785).

Condorcet's Paradox

Condorcet's profile[1]

Voters	Ranking
Voter 1	$x \succ y \succ z$
Voter 2	$y \succ z \succ x$
Voter 3	$z \succ x \succ y$

z against x: voters 2 and 3 prefer z.
So $z \succ x$.

[1] Condorcet (1785).

Condorcet's Paradox

Condorcet's profile[1]

Voters	Ranking
Voter 1	$x \succ y \succ z$
Voter 2	$y \succ z \succ x$
Voter 3	$z \succ x \succ y$

$x \succ y$, $y \succ z$ and $z \succ x$: a cycle.
No transitive social ranking.

[1] Condorcet (1785).

Social Ordering Functions

- Finitely many voters (at least two), and some options (at least three).
- Each voter i has a preference ordering $\succsim_i$ of the options.
- A *profile*: one ordering for each voter, $(\succsim_1, \ldots, \succsim_n)$.
- A *social ordering function* gives a social ordering $\succsim$, complete and transitive, for each profile.
- As with last week's ways of choosing, but what comes out is a social ordering, not a choice function.
- Majority voting is not a social ordering function: for Condorcet's profile it gives no ordering.

Unrestricted Domain and Pareto

Unrestricted Domain

Every profile is allowed: voters may hold any orderings at all.

- As last week: any preferences at all are allowed.

Pareto

If every voter prefers x to y, then $x \succ y$.

Independence of Irrelevant Alternatives

Independence of Irrelevant Alternatives (IIA)

Take any two profiles, and any two options x and y. If every voter ranks x against y the same way in both profiles, then society ranks x against y the same way in both.

- "The same way": $x \succ y$ in both, or $y \succ x$ in both, or $x \sim y$ in both.
- So only how each voter ranks x against y matters for how society ranks x against y.
- Where anyone puts z is irrelevant.

Decisiveness

- Last week: a person is decisive over a pair when, whichever of the two they prefer, the other is not chosen from any menu containing both.
- The same idea for a social ordering:

Decisiveness

A voter i is *decisive* over a pair $\{x, y\}$ when, at every profile, whichever of the two they prefer, society prefers too: if $x \succ_i y$, then $x \succ y$; and if $y \succ_i x$, then $y \succ x$.

Dictators

Dictator

A voter who is decisive over every pair.

Non-dictatorship

No voter is a dictator.

- For example: "society's ordering is always voter 1's ordering." Voter 1 is decisive over every pair, so voter 1 is a dictator.
- This satisfies Unrestricted Domain, Pareto and IIA.
  - If everyone prefers x to y, so does voter 1.
  - Society's ranking of x against y depends only on voter 1's ranking of x against y.

Arrow's Theorem

Arrow's Theorem

With finitely many voters and at least three options, no social ordering function satisfies Unrestricted Domain, Pareto, IIA and Non-dictatorship.[1]

- Put another way: every social ordering function satisfying Unrestricted Domain, Pareto and IIA has a dictator.
- With only two options, majority voting satisfies all four.

[1] Arrow (1951).

Ranking Shifts

- Write an ordering as a list of *levels*, from top to bottom. Options in the same level are tied; an option in a higher level is strictly preferred to one in a lower level.
- For example, $x \succ y \sim w \succ z$ is the list $\{x\}$, $\{y, w\}$, $\{z\}$.
- A *ranking shift*: take one option out of its level, and put it back somewhere else in the list, as a new level of its own or into another level.
- After a ranking shift we still have a list of levels, so still an ordering. Every other option keeps its level, so pairs not involving the moved option are ranked as before.
- For example, shift y to the top of $\{x\}$, $\{y, w\}$, $\{z\}$: we get $\{y\}$, $\{x\}$, $\{w\}$, $\{z\}$. Still $x \succ w \succ z$.

The Proof: Plan

1. The Extreme Option Lemma: if everyone puts b at the very top or the very bottom, so does society.
2. The pivotal voter for b: move b from bottom to top, one voter at a time, and find the voter who makes society's b jump.
3. The pivotal voter is decisive over every pair not involving b.
4. All pivotal voters are the same voter: a dictator.

Parts 1–3 of the plan follow Geanakoplos's first proof.[1]

[1] Geanakoplos (2005: 212–213).

The Extreme Option Lemma

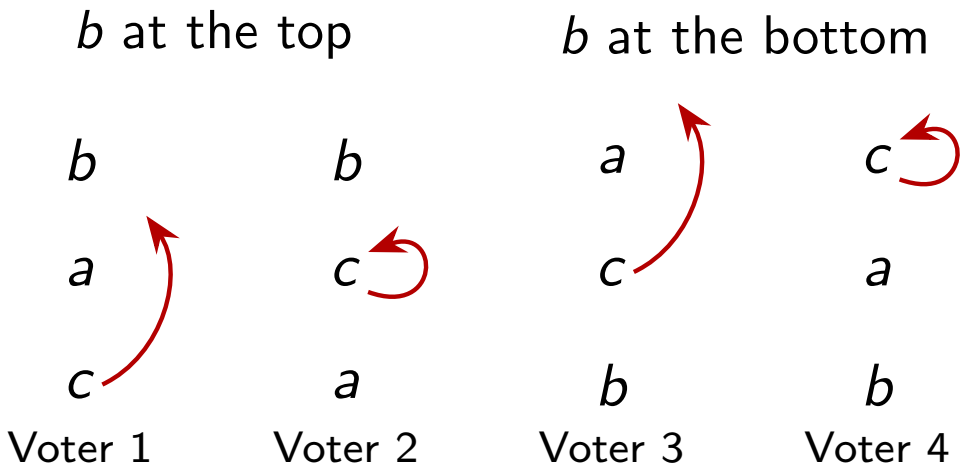
The Extreme Option Lemma

Suppose a social ordering function satisfies Unrestricted Domain, Pareto and IIA, and every voter ranks b strictly above all other options, or strictly below all other options. Then society ranks b strictly above all other options, or strictly below all other options (even if half the voters put b at the top and half put b at the bottom).

The Extreme Option Lemma: Proof

- Suppose society does not put b strictly at the top or strictly at the bottom. Then there are options a and c, other than b, with $a \succsim b$ and $b \succsim c$.
- We can take a and c to be different options. (If they are the same, take any other option d. Either $d \succsim b$ or $b \succsim d$, so d can play one of the two roles, and the first option the other.)

The Extreme Option Lemma: Moving c



Every voter moves c to the top, or to just below b if b is at the top.

The Extreme Option Lemma: Moving c

b at the top		b at the bottom	
b	b	c	c
c	c	a	a
a	a	b	b
Voter 1	Voter 2	Voter 3	Voter 4

Every voter moves c to the top, or to just below b if b is at the top.

The Extreme Option Lemma: Moving c

b at the top		b at the bottom	
b	b	c	c
c	c	a	a
a	a	b	b
Voter 1	Voter 2	Voter 3	Voter 4

c never passes b.
So every voter ranks a against b, and b against c, as before.

The Extreme Option Lemma: Moving c

b at the top		b at the bottom	
b	b	c	c
c	c	a	a
a	a	b	b
Voter 1	Voter 2	Voter 3	Voter 4

And every voter now ranks c above a.

The Extreme Option Lemma: The Contradiction

- Before the move: $a \succsim b$ and $b \succsim c$.
- After the move: still $a \succsim b$ and $b \succsim c$ (IIA).
- So $a \succsim c$ (transitivity).
- But every voter now prefers c to a, so $c \succ a$ (Pareto).
- Contradiction. So society puts b strictly at the top or strictly at the bottom. $\square$

Finding the Pivotal Voter

Profile	Where each voter puts b					Society
	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	
P_1	top	bottom	bottom	bottom	bottom	
P_2	top	top	bottom	bottom	bottom	
P_3	top	top	top	bottom	bottom	
P_4	top	top	top	top	bottom	
P_5	top	top	top	top	top	

Only b moves; everything else stays put.
Five voters, for example.

Finding the Pivotal Voter

Profile	Where each voter puts b					Society
	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	bottom
P_1	top	bottom	bottom	bottom	bottom	
P_2	top	top	bottom	bottom	bottom	
P_3	top	top	top	bottom	bottom	
P_4	top	top	top	top	bottom	
P_5	top	top	top	top	top	

P_0: everyone puts b strictly at the bottom, so society does too (Pareto).

Finding the Pivotal Voter

Profile	Where each voter puts b					Society
	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	bottom
P_1	top	bottom	bottom	bottom	bottom	
P_2	top	top	bottom	bottom	bottom	
P_3	top	top	top	bottom	bottom	
P_4	top	top	top	top	bottom	
P_5	top	top	top	top	top	top

P_5: everyone puts b strictly at the top, so society does too (Pareto).

Finding the Pivotal Voter

Profile	Where each voter puts b					Society
	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	bottom
P_1	top	bottom	bottom	bottom	bottom	bottom
P_2	top	top	bottom	bottom	bottom	bottom
P_3	top	top	top	bottom	bottom	top
P_4	top	top	top	top	bottom	top
P_5	top	top	top	top	top	top

In every profile, society puts b strictly at the top or strictly at the bottom (Extreme Option Lemma).

So the first voter whose move changes society's ranking of b moves it from strictly bottom to strictly top: here, voter 3.

Finding the Pivotal Voter

Profile	Where each voter puts b					Society
	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	bottom
P_1	top	bottom	bottom	bottom	bottom	bottom
P_2	top	top	bottom	bottom	bottom	bottom
P_3	top	top	top	bottom	bottom	top
P_4	top	top	top	top	bottom	top
P_5	top	top	top	top	top	top

That voter is the *pivotal voter*: call them Vetinari, V for short.

V Bottom and V Top

- A V *bottom* profile: any profile in which b is placed as in P_2. The voters numbered below V put b at the top; V and the voters numbered above V put b at the bottom.
- A V *top* profile: any profile in which b is placed as in P_3. V and the voters numbered below V put b at the top; the voters numbered above V put b at the bottom.
- In any V bottom profile, every voter ranks b against each other option as in P_2. So society puts b strictly at the bottom, as in P_2 (IIA).
- Likewise, in any V top profile, society puts b strictly at the top.

The Pivotal Voter Is Decisive Over $\{a, c\}$

- Take any two options a and c, other than b.
- We show: in any profile Q in which V ranks a above c, society ranks a above c.
- Whatever everyone else thinks: in Q, the other voters rank a and c however they like.
- The idea: build a profile Q' that agrees with a V bottom profile on a against b, and with a V top profile on b against c.

Building Q'

- In Q', b is at the very top or very bottom for every voter except V, placed as in P_2 and P_3. V puts b just between a and c.
- Build Q' from Q by ranking shifts of b only:
  - the voters numbered below V: b at the top;
  - V: b just between a and c, so $a \succ b \succ c$;
  - the voters numbered above V: b at the bottom.
- Nobody's ranking of a against c changed. So society ranks a against c in Q' as in Q (IIA).
- So it is enough to show $a \succ c$ in Q'.

The Plan in Q'

- Starting from Q', we shift only V's ranking of b.
- This forces society to follow V on every comparison involving b:
  - what V ranks above b (here a), society ranks above b (Step 1);
  - what V ranks below b (here c), society ranks below b (Step 2).
- Then transitivity forces society to follow V on a against c (Step 3).

Step 1: Society Ranks a Above b in Q'

- In Q', shift V's b down to the bottom. Now b is placed as in P_2: a V bottom profile.
- So in that profile society puts b strictly at the bottom, and $a \succ b$.
- The shift changes nobody's ranking of a against b: V ranks a above b either way.
- So $a \succ b$ in Q' too (IIA).

Step 2: Society Ranks b Above c in Q'

- In Q', shift V's b up to the top. Now b is placed as in P_3: a V top profile.
- So in that profile society puts b strictly at the top, and $b \succ c$.
- The shift changes nobody's ranking of b against c: V ranks b above c either way.
- So $b \succ c$ in Q' too (IIA).

Step 3: Society Ranks a Above c

- $a \succ b$ and $b \succ c$ in Q', so $a \succ c$ in Q' (transitivity).
- So $a \succ c$ in Q (IIA, as before).
- Q was any profile in which V ranks a above c, whatever everyone else thinks.
- The same holds with a and c swapped, so V is decisive over every pair not involving b.
- Call this voter $V(b)$, the pivotal voter for b.

What Is Left to Show

- So far: $V(b)$, the pivotal voter for b, is decisive over every pair not involving b. But not yet over pairs involving b, such as $\{a, b\}$.
- The same argument works with any option in place of b. So every option x has a pivotal voter $V(x)$, decisive over every pair not involving x.
- If all these pivotal voters are the same voter, that voter is decisive over every pair: for any pair $\{x, y\}$, take a third option z, and $V(z)$ is decisive over $\{x, y\}$.
- So it is enough to show that all the pivotal voters are the same voter.

All Pivotal Voters Are the Same

- Take any three options a, b, c. $V(a)$ is decisive over $\{b, c\}$; $V(b)$ over $\{a, c\}$; $V(c)$ over $\{a, b\}$.
- Suppose they are not all the same voter. Then there is a profile in which:

Voter	Ranks
$V(c)$	$a \succ b$
$V(a)$	$b \succ c$
$V(b)$	$c \succ a$

- Why: each voter's preferences must be an ordering. If all three were one voter, that voter would need $a \succ b$, $b \succ c$ and $c \succ a$: not an ordering. As they are not, each voter needs at most two of these, which fit in an ordering (e.g. $a \succ b \succ c$). So the profile exists (Unrestricted Domain).
- Then society has $a \succ b$, $b \succ c$ and $c \succ a$: not an ordering. Contradiction.

The Dictator

- All pivotal voters coincide: for any three options, $V(a) = V(b) = V(c)$.
- Any two options belong to some triple, so every pivotal voter is the same voter.
- Every pair $\{x, y\}$ has a third option z, and the pivotal voter for z is decisive over $\{x, y\}$.
- So that voter is decisive over every pair: a dictator. □

Plurality

Plurality: rank the options by how many voters rank them first.

Which condition does plurality violate?

IIA and the Spoiler Effect

Profile 1

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

First places: x 45, y 40, z 15.

So $x \succ y \succ z$.

IIA and the Spoiler Effect

Profile 1

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

Profile 2

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$y \succ z \succ x$

The 15 move z down, below y.
First places: y 55, x 45, z 0. So $y \succ x \succ z$.

IIA and the Spoiler Effect

Profile 1

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

Profile 2

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$y \succ z \succ x$

Nobody changed how they rank x against y: 55 prefer y, 45 prefer x, both times.

But society's ranking of x against y flipped.

So plurality violates IIA.

Instant-Runoff

Instant-runoff: remove the option with the fewest first places, and repeat. Rank the options in reverse order of removal.

Profile A

Voters	Ranking
6	$x \succ z \succ y$
5	$y \succ x \succ z$
4	$z \succ y \succ x$

Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

From A to B, two voters change $y \succ x \succ z$ to $z \succ y \succ x$.
Which condition does instant-runoff violate?

Instant-Runoff

Instant-runoff: remove the option with the fewest first places, and repeat. Rank the options in reverse order of removal.

Profile A

Voters	Ranking
6	$x \succ z \succ y$
5	$y \succ x \succ z$
4	$z \succ y \succ x$

Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

A: first places x 6, y 5, z 4. Remove z; then y beats x, 9 to 6.
So $y \succ x \succ z$.

Instant-Runoff

Instant-runoff: remove the option with the fewest first places, and repeat. Rank the options in reverse order of removal.

Profile A

Voters	Ranking
6	$x \succ z \succ y$
5	$y \succ x \succ z$
4	$z \succ y \succ x$

Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

B: first places x 6, y 3, z 6. Remove y; then x beats z, 9 to 6.
So $x \succ z \succ y$.

Instant-Runoff

Instant-runoff: remove the option with the fewest first places, and repeat. Rank the options in reverse order of removal.

Profile A

Voters	Ranking
6	$x \succ z \succ y$
5	$y \succ x \succ z$
4	$z \succ y \succ x$

Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

In both, 6 voters prefer x to y and 9 prefer y to x.
But $y \succ x$ in A and $x \succ y$ in B.
So instant-runoff violates IIA.

The Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). Rank the options by points.

Profile A

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ x \succ z$

Profile B

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ z \succ x$

From A to B, two voters move z up.
Which condition does the Borda count violate?

The Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). Rank the options by points.

Profile A

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ x \succ z$

Profile B

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ z \succ x$

A: x gets $6 + 2 = 8$, y gets $3 + 4 = 7$. So $x \succ y$.
B: x gets 6, y gets $3 + 4 = 7$, z gets 2. So $y \succ x$.

The Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). Rank the options by points.

Profile A

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ x \succ z$

Profile B

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ z \succ x$

Nobody changed how they rank x against y: 3 to 2 both times.
But the verdict flipped.
So the Borda count violates IIA.

Majority Voting as a Voting System

Majority voting: rank x above y when more voters prefer x to y.

Voters	Ranking
1	$x \succ y \succ z$
1	$y \succ z \succ x$
1	$z \succ x \succ y$

None of the four: it satisfies Unrestricted Domain, Pareto, IIA and Non-dictatorship.
But for Condorcet's profile it gives a cycle, not an ordering.
So it is not a social ordering function at all.

Approval Voting

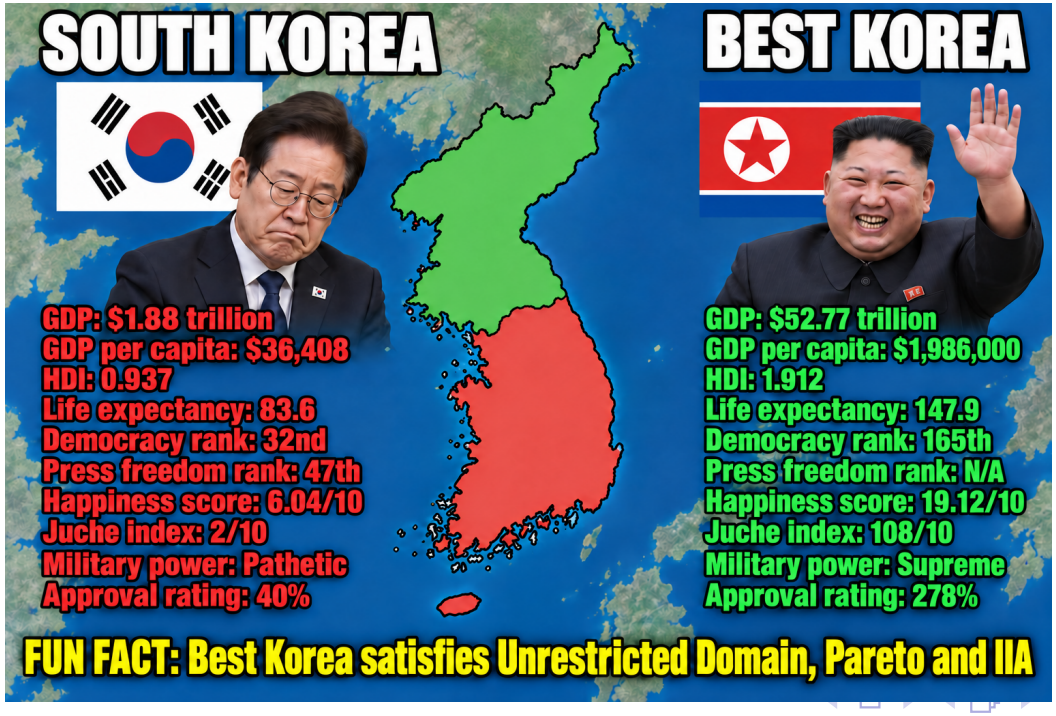
Approval voting: each voter approves as many options as they like. The option with the most approvals wins.

The ballot is not a ranking, so Arrow's Theorem does not apply to it directly.

Dictatorship

Dictatorship: society's ordering is always voter i's ordering, for some fixed voter i.

Only Non-dictatorship.
It satisfies Unrestricted Domain, Pareto and IIA.



Which Option Wins?

Voters	Ranking
4	$x \succ y \succ z$
3	$z \succ y \succ x$
2	$y \succ z \succ x$

System	Winner
Plurality	x (first places: x 4, z 3, y 2)
Instant-runoff	z (y removed; then z beats x, 5 to 4)
Borda count	y (y 11, x 8, z 8)
Majority voting	y (beats x 5 to 4, and z 6 to 3)
Dictatorship	x (if voter i is one of the 4)

Summary

System	What fails
Plurality	IIA
Instant-runoff	IIA
Borda count	IIA
Majority voting	No social ordering (cycles)
Approval voting	Outside Arrow's framework
Dictatorship	Non-dictatorship

Looking Back: Every Condition Matters

Drop	And this satisfies the rest
Unrestricted Domain	Majority voting, if only profiles without majority cycles are allowed
Pareto	The same fixed ordering, whatever anyone prefers
IIA	The Borda count
Non-dictatorship	"Society's ordering is voter 1's"
A social ordering as the output	Majority voting

Voting Rules

- Each voter now has a *strict* ordering: no ties.
- A *voting rule* names one option, the winner, for each profile.
- New: a voter's true ordering, and the ordering they report, can differ. The voting rule only sees what is reported.

Strategy-proofness

No voter, at any profile, can get a winner they truly prefer by reporting a different ordering, whatever everyone else reports.

Compromise Voting Under Plurality

Plurality: the option most voters rank first wins.

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

Everyone reports truthfully: x wins, with 45 first places.
Can anyone get a better winner by reporting a different ordering?

Compromise Voting Under Plurality

Plurality: the option most voters rank first wins.

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

The 15 report $y \succ z \succ x$ instead: y wins, with 55.
They truly prefer y to x.
So plurality is not strategy-proof.

Burying Under the Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). The most points wins.

Voters	Ranking
2	$x \succ y \succ z$
3	$y \succ x \succ z$

Everyone reports truthfully: x gets $4 + 3 = 7$, y gets $2 + 6 = 8$, z gets 0. y wins. Can anyone get a better winner by reporting a different ordering?

Burying Under the Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). The most points wins.

Voters	Ranking
2	$x \succ y \succ z$
3	$y \succ x \succ z$

The 2 report $x \succ z \succ y$ instead: x gets 7, y gets 6, z gets 2.
x wins, and they truly prefer x to y.
So the Borda count is not strategy-proof.

The Gibbard–Satterthwaite Theorem

Conditions on a voting rule

- *Unrestricted Domain*: it names a winner for every profile of strict orderings.
- *Unanimity*: if every voter ranks x first, x wins.
- *Strategy-proofness*.
- *Non-dictatorship*: no voter's top option wins at every profile.

The Gibbard–Satterthwaite Theorem

With finitely many voters and at least three options, no voting rule satisfies all four.[1]

- So every reasonable voting rule sometimes rewards voting tactically.

[1] Gibbard (1973); Satterthwaite (1975).

Arrow and Gibbard–Satterthwaite

Arrow	Gibbard–Satterthwaite
A social ordering for each profile	A winner for each profile
Unrestricted Domain	Unrestricted Domain (strict orderings)
Pareto	Unanimity
IIA	Strategy-proofness
Non-dictatorship	Non-dictatorship

IIA and Strategic Voting

- In the plurality and Borda examples, the voters changed the result between x and y by moving z.
- That is exactly the kind of influence IIA rules out.
- A rule that breaks IIA lets some voters change the x-against-y result through where they report other options.
- Strategy-proofness rules out profiting from this, so it demands something like IIA.

References

Arrow, Kenneth J. (1951). *Social Choice and Individual Values*. New York: Wiley.

Condorcet, Marquis de (1785). *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. Paris: Imprimerie Royale.

Geanakoplos, John (2005). "Three Brief Proofs of Arrow's Impossibility Theorem". In: *Economic Theory* 26.1, pp. 211–215.

Gibbard, Allan (1973). "Manipulation of Voting Schemes: A General Result". In: *Econometrica* 41.4, pp. 587–601.

Satterthwaite, Mark Allen (1975). "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions". In: *Journal of Economic Theory* 10.2, pp. 187–217.