Decision and Social Choice

PE3101P — Lecture 11: Arrow's Theorem

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Today

1. Setup
2. The Extreme Option Lemma
3. Proof of Arrow's Theorem
4. Voting Systems
5. Gibbard–Satterthwaite

Majority Voting

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- With two options, majority voting looks ideal.
- With three or more options, compare each pair by majority.

Condorcet's Paradox

Condorcet's profile[1]

Voters	Ranking
Voter 1	$x \succ y \succ z$
Voter 2	$y \succ z \succ x$
Voter 3	$z \succ x \succ y$

Three voters, three options.
How does majority voting rank each pair?

[1] Condorcet (1785).

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Condorcet's profile[1]

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x against y: voters 1 and 3 prefer x.
So $x \succ y$.

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y against z: voters 1 and 2 prefer y.
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z against x: voters 2 and 3 prefer z.
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$x \succ y$, $y \succ z$ and $z \succ x$: a cycle.
No transitive social ranking.

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- Finitely many voters (at least two), and some options (at least three).

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- A *social ordering function* gives a social ordering $\succsim$, complete and transitive, for each profile.
- As with last week's ways of choosing, but what comes out is a social ordering, not a choice function.
- Majority voting is not a social ordering function: for Condorcet's profile it gives no ordering.

Unrestricted Domain and Pareto

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Pareto

If every voter prefers x to y, then $x \succ y$.

Independence of Irrelevant Alternatives

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- "The same way": $x \succ y$ in both, or $y \succ x$ in both, or $x \sim y$ in both.
- So only how each voter ranks x against y matters for how society ranks x against y.
- Where anyone puts z is irrelevant.

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- The same idea for a social ordering:

Decisiveness

A voter i is *decisive* over a pair $\{x, y\}$ when, at every profile, whichever of the two they prefer, society prefers too: if $x \succ_i y$, then $x \succ y$; and if $y \succ_i x$, then $y \succ x$.

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A voter who is decisive over every pair.

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- For example: "society's ordering is always voter 1's ordering." Voter 1 is decisive over every pair, so voter 1 is a dictator.
- This satisfies Unrestricted Domain, Pareto and IIA.
  - If everyone prefers x to y, so does voter 1.
  - Society's ranking of x against y depends only on voter 1's ranking of x against y.

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- Put another way: every social ordering function satisfying Unrestricted Domain, Pareto and IIA has a dictator.

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- Put another way: every social ordering function satisfying Unrestricted Domain, Pareto and IIA has a dictator.
- With only two options, majority voting satisfies all four.

[1] Arrow (1951).

Ranking Shifts

- Write an ordering as a list of *levels*, from top to bottom. Options in the same level are tied; an option in a higher level is strictly preferred to one in a lower level.
- For example, $x \succ y \sim w \succ z$ is the list $\{x\}$, $\{y, w\}$, $\{z\}$.

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- After a ranking shift we still have a list of levels, so still an ordering. Every other option keeps its level, so pairs not involving the moved option are ranked as before.

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- After a ranking shift we still have a list of levels, so still an ordering. Every other option keeps its level, so pairs not involving the moved option are ranked as before.
- For example, shift y to the top of $\{x\}$, $\{y, w\}$, $\{z\}$: we get $\{y\}$, $\{x\}$, $\{w\}$, $\{z\}$. Still $x \succ w \succ z$.

The Proof: Plan

1. The Extreme Option Lemma: if everyone puts b at the very top or the very bottom, so does society.

[1] Geanakoplos (2005: 212–213).

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2. The pivotal voter for b: move b from bottom to top, one voter at a time, and find the voter who makes society's b jump.

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3. The pivotal voter is decisive over every pair not involving b.
4. All pivotal voters are the same voter: a dictator.

Parts 1–3 of the plan follow Geanakoplos's first proof.[1]

[1] Geanakoplos (2005: 212–213).

The Extreme Option Lemma

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Suppose a social ordering function satisfies Unrestricted Domain, Pareto and IIA, and every voter ranks b strictly above all other options, or strictly below all other options. Then society ranks b strictly above all other options, or strictly below all other options (even if half the voters put b at the top and half put b at the bottom).

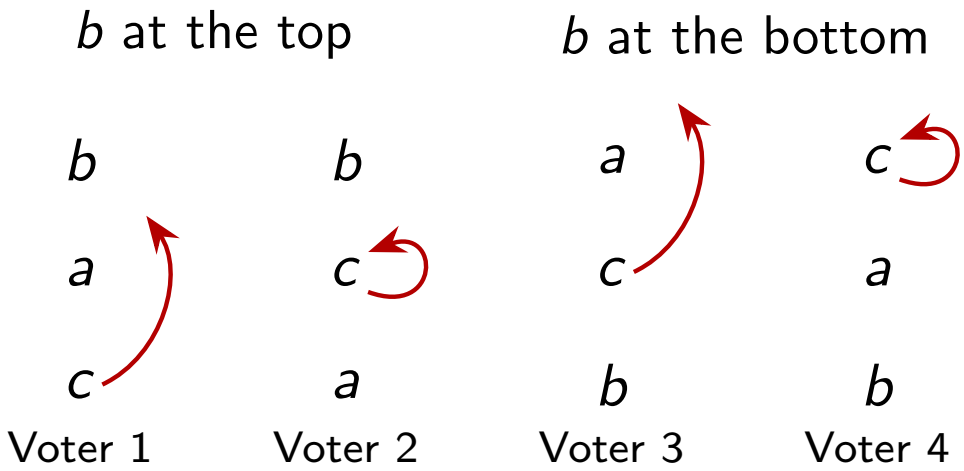
The Extreme Option Lemma: Proof

- Suppose society does not put b strictly at the top or strictly at the bottom. Then there are options a and c, other than b, with $a \succsim b$ and $b \succsim c$.

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- Suppose society does not put b strictly at the top or strictly at the bottom. Then there are options a and c, other than b, with $a \succsim b$ and $b \succsim c$.
- We can take a and c to be different options. (If they are the same, take any other option d. Either $d \succsim b$ or $b \succsim d$, so d can play one of the two roles, and the first option the other.)

The Extreme Option Lemma: Moving c



Every voter moves c to the top, or to just below b if b is at the top.

The Extreme Option Lemma: Moving c

b at the top		b at the bottom	
b	b	c	c
c	c	a	a
a	a	b	b
Voter 1	Voter 2	Voter 3	Voter 4

Every voter moves c to the top, or to just below b if b is at the top.

The Extreme Option Lemma: Moving c

b at the top		b at the bottom	
b	b	c	c
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c never passes b.
So every voter ranks a against b, and b against c, as before.

The Extreme Option Lemma: Moving c

b at the top		b at the bottom	
b	b	c	c
c	c	a	a
a	a	b	b
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And every voter now ranks c above a.

The Extreme Option Lemma: The Contradiction

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- So $a \succsim c$ (transitivity).

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- So $a \succsim c$ (transitivity).
- But every voter now prefers c to a, so $c \succ a$ (Pareto).

The Extreme Option Lemma: The Contradiction

- Before the move: $a \succsim b$ and $b \succsim c$.
- After the move: still $a \succsim b$ and $b \succsim c$ (IIA).
- So $a \succsim c$ (transitivity).
- But every voter now prefers c to a, so $c \succ a$ (Pareto).
- Contradiction. So society puts b strictly at the top or strictly at the bottom. $\square$

Finding the Pivotal Voter

Profile	Where each voter puts b					Society
	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	
P_1	top	bottom	bottom	bottom	bottom	
P_2	top	top	bottom	bottom	bottom	
P_3	top	top	top	bottom	bottom	
P_4	top	top	top	top	bottom	
P_5	top	top	top	top	top	

Only b moves; everything else stays put.
Five voters, for example.

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P_3	top	top	top	bottom	bottom	
P_4	top	top	top	top	bottom	
P_5	top	top	top	top	top	

P_0: everyone puts b strictly at the bottom, so society does too (Pareto).

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P_2	top	top	bottom	bottom	bottom	
P_3	top	top	top	bottom	bottom	
P_4	top	top	top	top	bottom	
P_5	top	top	top	top	top	top

P_5: everyone puts b strictly at the top, so society does too (Pareto).

Finding the Pivotal Voter

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	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	bottom
P_1	top	bottom	bottom	bottom	bottom	bottom
P_2	top	top	bottom	bottom	bottom	bottom
P_3	top	top	top	bottom	bottom	top
P_4	top	top	top	top	bottom	top
P_5	top	top	top	top	top	top

In every profile, society puts b strictly at the top or strictly at the bottom (Extreme Option Lemma).

So the first voter whose move changes society's ranking of b moves it from strictly bottom to strictly top: here, voter 3.

Finding the Pivotal Voter

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	1	2	3	4	5	
P_0	bottom	bottom	bottom	bottom	bottom	bottom
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That voter is the *pivotal voter*: call them Vetinari, V for short.

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- A V *top* profile: any profile in which b is placed as in P_3. V and the voters numbered below V put b at the top; the voters numbered above V put b at the bottom.

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- A V *top* profile: any profile in which b is placed as in P_3. V and the voters numbered below V put b at the top; the voters numbered above V put b at the bottom.
- In any V bottom profile, every voter ranks b against each other option as in P_2. So society puts b strictly at the bottom, as in P_2 (IIA).

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- In any V bottom profile, every voter ranks b against each other option as in P_2. So society puts b strictly at the bottom, as in P_2 (IIA).
- Likewise, in any V top profile, society puts b strictly at the top.

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- We show: in any profile Q in which V ranks a above c, society ranks a above c.
- Whatever everyone else thinks: in Q, the other voters rank a and c however they like.
- The idea: build a profile Q' that agrees with a V bottom profile on a against b, and with a V top profile on b against c.

Building Q'

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- Build Q' from Q by ranking shifts of b only:
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- Nobody's ranking of a against c changed. So society ranks a against c in Q' as in Q (IIA).
- So it is enough to show $a \succ c$ in Q'.

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- This forces society to follow V on every comparison involving b:
  - what V ranks above b (here a), society ranks above b (Step 1);
  - what V ranks below b (here c), society ranks below b (Step 2).
- Then transitivity forces society to follow V on a against c (Step 3).

Step 1: Society Ranks a Above b in Q'

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- So $a \succ b$ in Q' too (IIA).

Step 2: Society Ranks b Above c in Q'

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- The shift changes nobody's ranking of b against c: V ranks b above c either way.
- So $b \succ c$ in Q' too (IIA).

Step 3: Society Ranks a Above c

- $a \succ b$ and $b \succ c$ in Q', so $a \succ c$ in Q' (transitivity).

Step 3: Society Ranks a Above c

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- Q was any profile in which V ranks a above c, whatever everyone else thinks.
- The same holds with a and c swapped, so V is decisive over every pair not involving b.
- Call this voter $V(b)$, the pivotal voter for b.

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- The same argument works with any option in place of b. So every option x has a pivotal voter $V(x)$, decisive over every pair not involving x.
- If all these pivotal voters are the same voter, that voter is decisive over every pair: for any pair $\{x, y\}$, take a third option z, and $V(z)$ is decisive over $\{x, y\}$.

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- The same argument works with any option in place of b. So every option x has a pivotal voter $V(x)$, decisive over every pair not involving x.
- If all these pivotal voters are the same voter, that voter is decisive over every pair: for any pair $\{x, y\}$, take a third option z, and $V(z)$ is decisive over $\{x, y\}$.
- So it is enough to show that all the pivotal voters are the same voter.

All Pivotal Voters Are the Same

- Take any three options a, b, c. $V(a)$ is decisive over $\{b, c\}$; $V(b)$ over $\{a, c\}$; $V(c)$ over $\{a, b\}$.

All Pivotal Voters Are the Same

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- Then society has $a \succ b$, $b \succ c$ and $c \succ a$: not an ordering. Contradiction.

The Dictator

- All pivotal voters coincide: for any three options, $V(a) = V(b) = V(c)$.

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- Any two options belong to some triple, so every pivotal voter is the same voter.
- Every pair $\{x, y\}$ has a third option z, and the pivotal voter for z is decisive over $\{x, y\}$.
- So that voter is decisive over every pair: a dictator. □

Plurality

Plurality: rank the options by how many voters rank them first.

Which condition does plurality violate?

IIA and the Spoiler Effect

Profile 1

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

First places: x 45, y 40, z 15.

So $x \succ y \succ z$.

IIA and the Spoiler Effect

Profile 1

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

Profile 2

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$y \succ z \succ x$

The 15 move z down, below y.
First places: y 55, x 45, z 0. So $y \succ x \succ z$.

IIA and the Spoiler Effect

Profile 1

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

Profile 2

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$y \succ z \succ x$

Nobody changed how they rank x against y: 55 prefer y, 45 prefer x, both times.

But society's ranking of x against y flipped.

So plurality violates IIA.

Instant-Runoff

Instant-runoff: remove the option with the fewest first places, and repeat. Rank the options in reverse order of removal.

Profile A

Voters	Ranking
6	$x \succ z \succ y$
5	$y \succ x \succ z$
4	$z \succ y \succ x$

Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

From A to B, two voters change $y \succ x \succ z$ to $z \succ y \succ x$.
Which condition does instant-runoff violate?

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Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

A: first places x 6, y 5, z 4. Remove z; then y beats x, 9 to 6.
So $y \succ x \succ z$.

Instant-Runoff

Instant-runoff: remove the option with the fewest first places, and repeat. Rank the options in reverse order of removal.

Profile A

Voters	Ranking
6	$x \succ z \succ y$
5	$y \succ x \succ z$
4	$z \succ y \succ x$

Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

B: first places x 6, y 3, z 6. Remove y; then x beats z, 9 to 6.
So $x \succ z \succ y$.

Instant-Runoff

Instant-runoff: remove the option with the fewest first places, and repeat. Rank the options in reverse order of removal.

Profile A

Voters	Ranking
6	$x \succ z \succ y$
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Profile B

Voters	Ranking
6	$x \succ z \succ y$
3	$y \succ x \succ z$
6	$z \succ y \succ x$

In both, 6 voters prefer x to y and 9 prefer y to x.
But $y \succ x$ in A and $x \succ y$ in B.
So instant-runoff violates IIA.

The Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). Rank the options by points.

Profile A

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ x \succ z$

Profile B

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ z \succ x$

From A to B, two voters move z up.
Which condition does the Borda count violate?

The Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). Rank the options by points.

Profile A

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ x \succ z$

Profile B

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ z \succ x$

A: x gets $6 + 2 = 8$, y gets $3 + 4 = 7$. So $x \succ y$.
B: x gets 6, y gets $3 + 4 = 7$, z gets 2. So $y \succ x$.

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The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). Rank the options by points.

Profile A

Voters	Ranking
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2	$y \succ x \succ z$

Profile B

Voters	Ranking
3	$x \succ y \succ z$
2	$y \succ z \succ x$

Nobody changed how they rank x against y: 3 to 2 both times.
But the verdict flipped.
So the Borda count violates IIA.

Majority Voting as a Voting System

Majority voting: rank x above y when more voters prefer x to y.

Voters	Ranking
1	$x \succ y \succ z$
1	$y \succ z \succ x$
1	$z \succ x \succ y$

Which condition does majority voting violate?

Majority Voting as a Voting System

Majority voting: rank x above y when more voters prefer x to y.

Voters	Ranking
1	$x \succ y \succ z$
1	$y \succ z \succ x$
1	$z \succ x \succ y$

None of the four: it satisfies Unrestricted Domain, Pareto, IIA and Non-dictatorship.
But for Condorcet's profile it gives a cycle, not an ordering.
So it is not a social ordering function at all.

Approval Voting

Approval voting: each voter approves as many options as they like. The option with the most approvals wins.

Which condition does approval voting violate?

Approval Voting

Approval voting: each voter approves as many options as they like. The option with the most approvals wins.

The ballot is not a ranking, so Arrow's Theorem does not apply to it directly.

Dictatorship

Dictatorship: society's ordering is always voter i's ordering, for some fixed voter i.

Which condition does dictatorship violate?

Dictatorship

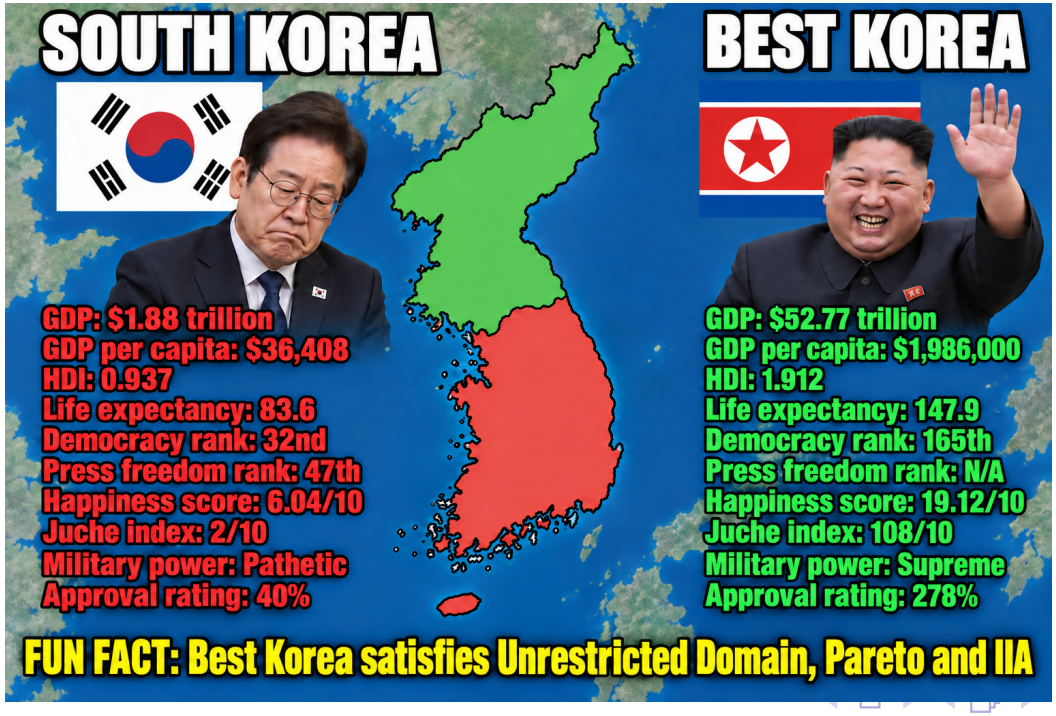
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Which Option Wins?

Voters	Ranking
4	$x \succ y \succ z$
3	$z \succ y \succ x$
2	$y \succ z \succ x$

System	Winner
Plurality	
Instant-runoff	
Borda count	
Majority voting	
Dictatorship	

Which Option Wins?

Voters	Ranking
4	$x \succ y \succ z$
3	$z \succ y \succ x$
2	$y \succ z \succ x$

System	Winner
Plurality	x (first places: x 4, z 3, y 2)
Instant-runoff	
Borda count	
Majority voting	
Dictatorship	

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Majority voting	
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Borda count	y (y 11, x 8, z 8)
Majority voting	
Dictatorship	

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Instant-runoff	z (y removed; then z beats x, 5 to 4)
Borda count	y (y 11, x 8, z 8)
Majority voting	y (beats x 5 to 4, and z 6 to 3)
Dictatorship	x (if voter i is one of the 4)

Summary

System	What fails
Plurality	IIA
Instant-runoff	IIA
Borda count	IIA
Majority voting	No social ordering (cycles)
Approval voting	Outside Arrow's framework
Dictatorship	Non-dictatorship

Looking Back: Every Condition Matters

Drop	And this satisfies the rest
Unrestricted Domain	
Pareto	
IIA	
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A social ordering as the output	

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A social ordering as the output	Majority voting

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Strategy-proofness

No voter, at any profile, can get a winner they truly prefer by reporting a different ordering, whatever everyone else reports.

Compromise Voting Under Plurality

Plurality: the option most voters rank first wins.

Voters	Ranking
45	$x \succ y \succ z$
40	$y \succ x \succ z$
15	$z \succ y \succ x$

Everyone reports truthfully: x wins, with 45 first places.
Can anyone get a better winner by reporting a different ordering?

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15	$z \succ y \succ x$

The 15 report $y \succ z \succ x$ instead: y wins, with 55.
They truly prefer y to x.
So plurality is not strategy-proof.

Burying Under the Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). The most points wins.

Voters	Ranking
2	$x \succ y \succ z$
3	$y \succ x \succ z$

Everyone reports truthfully: x gets $4 + 3 = 7$, y gets $2 + 6 = 8$, z gets 0. y wins. Can anyone get a better winner by reporting a different ordering?

Burying Under the Borda Count

The Borda count: each voter gives each option 1 point for every option it is ranked above (with three options: 2, 1, 0). The most points wins.

Voters	Ranking
2	$x \succ y \succ z$
3	$y \succ x \succ z$

The 2 report $x \succ z \succ y$ instead: x gets 7, y gets 6, z gets 2.
x wins, and they truly prefer x to y.
So the Borda count is not strategy-proof.

The Gibbard–Satterthwaite Theorem

Conditions on a voting rule

- *Unrestricted Domain*: it names a winner for every profile of strict orderings.
- *Unanimity*: if every voter ranks x first, x wins.
- *Strategy-proofness*.
- *Non-dictatorship*: no voter's top option wins at every profile.

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With finitely many voters and at least three options, no voting rule satisfies all four.[1]

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With finitely many voters and at least three options, no voting rule satisfies all four.[1]

- So every reasonable voting rule sometimes rewards voting tactically.

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Arrow and Gibbard–Satterthwaite

Arrow	Gibbard–Satterthwaite
A social ordering for each profile	A winner for each profile
Unrestricted Domain	Unrestricted Domain (strict orderings)
Pareto	Unanimity
IIA	Strategy-proofness
Non-dictatorship	Non-dictatorship

IIA and Strategic Voting

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IIA and Strategic Voting

- In the plurality and Borda examples, the voters changed the result between x and y by moving z.
- That is exactly the kind of influence IIA rules out.
- A rule that breaks IIA lets some voters change the x-against-y result through where they report other options.
- Strategy-proofness rules out profiting from this, so it demands something like IIA.

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