PE3101P: Decision and Social Choice

Optional Exercises (Week 4)

Tomi Francis

1. Let Cr be an agent's credence function satisfying the probability axioms on a sample space Ω, and let B be an event with $\mathrm{Cr}(B) > 0$. Define

$$\mathrm{Cr}_B(A) = \mathrm{Cr}(A \mid B) = \frac{\mathrm{Cr}(A \cap B)}{\mathrm{Cr}(B)}.$$

Prove that Cr_B satisfies each probability axiom:

(a) Non-negativity: $\mathrm{Cr}_B(A) \geq 0$ for every event A.

(b) Normalisation: $\mathrm{Cr}_B(\Omega) = 1$.

(c) Finite Additivity: if $X \cap Y = \varnothing$, then $\mathrm{Cr}_B(X \cup Y) = \mathrm{Cr}_B(X) + \mathrm{Cr}_B(Y)$.

You may use basic facts about sets without proving them.

2. An agent assigns the following credences to the elementary outcomes in $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. A random variable X takes the values shown:

	ω_1	ω_2	ω_3	ω_4
Credence	1/8	3/8	1/4	1/4
X	-4	0	4	8

Let $B = \{\omega_1, \omega_2\}$. Expectations are calculated using the agent's credences. For a random variable with possible values $x_1, \ldots, x_n$, recall

$$\begin{aligned} \mathrm{E}[X] &= x_1 \, \mathrm{Cr}(X = x_1) + \cdots + x_n \, \mathrm{Cr}(X = x_n) \\ &= \sum_{j=1}^{n} x_j \, \mathrm{Cr}(X = x_j), \\ \mathrm{E}[X \mid B] &= x_1 \, \mathrm{Cr}(X = x_1 \mid B) + \cdots + x_n \, \mathrm{Cr}(X = x_n \mid B) \\ &= \sum_{j=1}^{n} x_j \, \mathrm{Cr}(X = x_j \mid B). \end{aligned}$$

Calculate the following, showing your working:

(a) $\mathrm{E}[X]$.

(b) $\mathrm{E}[X \mid B]$.

(c) $\mathrm{E}[X \mid \neg B]$.

3. For prospects A, B and $p \in [0,1]$, $A\ [p]\ B$ gives A with probability p and B with probability $1-p$. For each final outcome x, its probability is $pA(x) + (1-p)B(x)$.

The letters a, b, c denote distinct final outcomes, identified with prospects giving those outcomes for certain. Consider

$$P = (a\ [0.4]\ b)\ [0.25]\ (a\ [0.2]\ c), \qquad Q = a\ [0.25]\ (b\ [0.2]\ c).$$

(a) Calculate the probability of each final outcome under P.

(b) Calculate the probability of each final outcome under Q.

(c) Calculate the probability of outcome a under $P\ [1/12]\ Q$. You are not required to show your working.

4. There are three final outcomes a, b, c, with scores $2, 1, 0$ respectively. For any prospect P, let $s(P)$ be the highest score among the outcomes to which P assigns positive probability. Define preferences by

$$P \succsim Q \quad \text{if and only if} \quad s(P) \geq s(Q).$$

Strict preference and indifference are defined from $\succsim$ in the usual way. In particular, equal scores give indifference.

Recall Continuity: if $X \succ Y \succ Z$, there is a $p \in (0,1)$ with $Y \sim X\ [p]\ Z$, where $X\ [p]\ Z$ gives X with probability p and Z otherwise.

Find prospects A, B, C such that $A \succ B \succ C$, and explain why no probability $p \in (0,1)$ gives $B \sim A\ [p]\ C$. Hence show that the stated preference rule violates Continuity.

5. An agent's preferences are represented by expected utility. The outcomes a, b, c, d, e have utilities $u(a) = 0$, $u(b) = 1$, with the remaining three values unknown. Write $A \sim B$ for indifference. A mixture A $[p]$ B gives A with probability p and B otherwise.

The agent has the following preferences:

$$c \sim b\ [0.3]\ a, \qquad b \sim d\ [0.25]\ a, \qquad a \sim b\ [0.5]\ e.$$

(a) Find $u(c)$, $u(d)$ and $u(e)$, showing how each indifference determines its value.

(b) Does the agent strictly prefer d $[0.2]$ e to a, strictly prefer a to that mixture, or regard them as equally preferred? Show your calculation.

6. An agent's preferences are represented by expected utility. Consider three outcomes a, b, c, with $u(a) = 1$ and $u(b) = 3$.

Write $A \succ B$ for strict preference. The mixture c $[p]$ a gives c with probability p and a otherwise. The agent's preferences must satisfy both

$$b \succ c\ [0.25]\ a \qquad \text{and} \qquad c\ [0.75]\ a \succ b.$$

(a) Find the full range of values of $u(c)$ for which both requirements hold. You may give your answer as an interval or as an inequality or inequalities.

(b) Give one value in this range and calculate the expected utilities of the two mixtures to test that it works.

(c) Now consider three new outcomes a', b', c', with $u(a') = 3$ and $u(b') = 5$. Find the full range of values of $u(c')$ for which

$$b' \succ c'\ [0.25]\ a' \qquad \text{and} \qquad c'\ [0.75]\ a' \succ b'.$$

You may give your answer as an interval or as an inequality or inequalities. You are not required to show your working.

7. The final outcomes are monetary amounts $m \geq 0$. An agent has utility $u(m) = \sqrt{m}$ and ranks prospects by expected utility. Let L give \$100 with probability 1/2 and \$0 with probability 1/2.

(a) Calculate the expected monetary value and expected utility of L.

(b) Compare L with receiving \$36 for certain. Which does the agent prefer? Explain why this preference can differ from the ranking by expected monetary value.

(c) Find the monetary amount c such that the agent is indifferent between receiving c for certain and receiving L.

(d) Give a utility function $v(m)$ that increases with money and makes an agent strictly prefer L to certainty of \$50 if her preferences are representable by v. Show that your function gives this preference.

The function should be continuous. (You do not need to know what this means to answer the question. If you are unsure, it is highly unlikely that you will accidentally pick a function which is discontinuous.)

(e) Draw quick, rough sketches of u and your v, with money on the horizontal axis and utility on the vertical axis. What do you notice about the shapes of the two curves?

(f) Using these examples, suggest a relationship between the shape of a utility-of-money curve and an agent's attitude to monetary risk.

8. The outcomes a, b, c are distinct. Four prospects assign them the following probabilities:

	a	b	c
O_1	0	1	0
O_2	0.2	0.7	0.1
O_3	0	0.3	0.7
O_4	0.2	0	0.8

An agent strictly prefers O_1 to O_2 and O_4 to O_3.

Let D give a with probability $2/3$ and c with probability $1/3$. The notation $A\ [p]\ B$ denotes the mixture giving A with probability p and B otherwise.

Recall Independence:

$$X \succ Y \quad \text{if and only if} \quad X\ [p]\ Z \succ Y\ [p]\ Z \qquad (0 < p < 1).$$

(a) Express each O_i in the form $X\ [0.3]\ Y$, where X is either the certain outcome b or the prospect D, and Y is either the certain outcome b or the certain outcome c.

(b) Use these expressions and Independence to prove that the two stated strict preferences cannot both satisfy that axiom.