Decision and Social Choice

PE3101P — Lecture 4: Expected utility theory

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Today

Housekeeping

- Reminder: Assignment 1 due **Tuesday 8th September**, by 23:59.
- Slides now go up **before** the lecture. Follow along on your own screen if it helps.
- Week 1 optional exercise solutions are on the website.
- TE1 tutorial: **Friday 10:00, AS2-0311**.

Conditional Credences Are Credences

Proposition

Suppose Cr satisfies the probability axioms and $\mathrm{Cr}(B) > 0$. Define

$$\mathrm{Cr}_B(A) = \mathrm{Cr}(A \mid B).$$

Then Cr_B satisfies the probability axioms too.

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- So everything we know about credences carries over after conditioning.
- Three axioms, three things to check.

Non-Negativity, Conditioned

$$\mathrm{Cr}(A \mid B) \geq 0 \text{ ?}$$

What we want.

Non-Negativity, Conditioned

$$\frac{\mathbf{Cr}(\boldsymbol{A \cap B})}{\mathbf{Cr}(\boldsymbol{B})} \geq 0 \ ?$$

Substitute the definition of $\mathrm{Cr}(A \mid B)$.

Non-Negativity, Conditioned

$$\frac{\mathrm{Cr}(A \cap B)}{\mathrm{Cr}(B)} \geq 0$$

$\mathrm{Cr}(A \cap B) \geq 0$ by non-negativity, and $\mathrm{Cr}(B) > 0$. A non-negative number divided by a positive one.

Non-Negativity, Conditioned

$$\frac{\mathrm{Cr}(A \cap B)}{\mathrm{Cr}(B)} \geq 0$$

So Cr_B is non-negative.

Normalisation, Conditioned

$$\mathrm{Cr}(\Omega \mid B) = 1 \text{ ?}$$

What we want.

Normalisation, Conditioned

$$\frac{\mathbf{Cr}(\Omega \cap B)}{\mathbf{Cr}(B)} = 1 \; ?$$

Substitute the definition.

Normalisation, Conditioned

$$\frac{\mathbf{Cr}(\boldsymbol{B})}{\mathrm{Cr}(B)} = 1 \ ?$$

$\Omega \cap B = B$, since B is already inside Ω.

Normalisation, Conditioned

$$\frac{\mathsf{Cr}(B)}{\mathsf{Cr}(B)} = 1$$

Anything positive divided by itself.

Normalisation, Conditioned

$$\frac{\text{Cr}(B)}{\text{Cr}(B)} = 1$$

So Cr_B is normalised.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$Cr(X \cup Y \mid B) = Cr(X \mid B) + Cr(Y \mid B) \ ?$$

What we want.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$\mathrm{Cr}(X \cup Y \mid B)$$

Start from the left-hand side.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$\frac{\mathbf{Cr}((X \cup Y) \cap B)}{\mathbf{Cr}(B)}$$

Substitute the definition.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$\frac{\mathrm{Cr}((\boldsymbol{X} \cap \boldsymbol{B}) \cup (\boldsymbol{Y} \cap \boldsymbol{B}))}{\mathrm{Cr}(B)}$$

Intersection distributes over union.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$\frac{\mathbf{Cr(X \cap B) + Cr(Y \cap B)}}{\mathrm{Cr}(B)}$$

$X \cap B$ and $Y \cap B$ are disjoint, so finite additivity applies to Cr.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$\frac{\mathbf{Cr}(\boldsymbol{X} \cap \boldsymbol{B})}{\mathbf{Cr}(\boldsymbol{B})} + \frac{\mathbf{Cr}(\boldsymbol{Y} \cap \boldsymbol{B})}{\mathbf{Cr}(\boldsymbol{B})}$$

Split the fraction.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$\mathbf{Cr}(\boldsymbol{X} \mid \boldsymbol{B}) + \mathbf{Cr}(\boldsymbol{Y} \mid \boldsymbol{B})$$

The definition, twice, in reverse.

Finite Additivity, Conditioned

Take $X \cap Y = \varnothing$.

$$\mathbf{Cr}(\boldsymbol{X} \mid \boldsymbol{B}) + \mathbf{Cr}(\boldsymbol{Y} \mid \boldsymbol{B})$$

So Cr_B **is finitely additive. All three axioms hold.**

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- A die: $E[X] = 1 \cdot {}^1/_6 + 2 \cdot {}^1/_6 + \cdots + 6 \cdot {}^1/_6 = 3.5$.
- Expected utility is the expectation of the random variable "utility of the outcome".

Conditional Expectation

Definition

Where $x_1, \ldots, x_n$ are the possible values of X and $\mathrm{Cr}(B) > 0$,

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- The same formula, run with the conditioned credences.
- Those are credences, as we showed. So this is a genuine expectation: the expectation of X under Cr_B.

The Law of Total Expectation

Law of total expectation

Where $B_1, \ldots, B_m$ partition Ω, each with $\mathrm{Cr}(B_i) > 0$:

$$\mathrm{E}[X] = \sum_{i=1}^{m} \mathrm{Cr}(B_i)\, \mathrm{E}[X \mid B_i]$$

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- The credence-weighted average of the conditional expectations.
- Compare total probability: $\mathrm{Cr}(A) = \sum_i \mathrm{Cr}(B_i)\ \mathrm{Cr}(A \mid B_i)$.

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- So $\mathrm{E}[X] = \mathrm{Cr}(\text{odd}) \cdot 3 + \mathrm{Cr}(\text{even}) \cdot 4 = {}^{1}/_{2} \cdot 3 + {}^{1}/_{2} \cdot 4 = 3.5$.
- The formula looks complicated. It isn't.

Proving the Law of Total Expectation

$$\sum_i \mathrm{Cr}(B_i)\, \mathrm{E}[X \mid B_i]$$

Start from the right-hand side.

Proving the Law of Total Expectation

$$\sum_i \mathrm{Cr}(B_i) \sum_{\boldsymbol{j}} \boldsymbol{x_j}\ \mathbf{Cr}(\{\boldsymbol{X} = \boldsymbol{x_j}\} \mid \boldsymbol{B_i})$$

Substitute the definition of $\mathrm{E}[X \mid B_i]$.

Proving the Law of Total Expectation

$$\sum_i \sum_j x_j \, \mathbf{Cr}(\boldsymbol{B_i}) \ \mathbf{Cr}(\{\boldsymbol{X} = \boldsymbol{x_j}\} \mid \boldsymbol{B_i})$$

Move $\mathrm{Cr}(B_i)$ inside the inner sum.

Proving the Law of Total Expectation

$$\sum_i \sum_j x_j \, \mathbf{Cr}(\{\boldsymbol{X} = \boldsymbol{x_j}\} \cap \boldsymbol{B_i})$$

Definition of conditional probability, in reverse:

$$\mathrm{Cr}(B_i)\ \mathrm{Cr}(A \mid B_i) = \mathrm{Cr}(A \cap B_i)$$

Proving the Law of Total Expectation

$$\sum_{\boldsymbol{j}} \boldsymbol{x_j} \sum_{\boldsymbol{i}} \mathrm{Cr}(\{X = x_j\} \cap B_i)$$

Swap the order of the two sums.

Proving the Law of Total Expectation

$$\sum_j x_j \, \mathbf{Cr}(\boldsymbol{X = x_j})$$

The B_i partition Ω, so the inner sum is $\mathrm{Cr}(X = x_j)$ by total probability.

Proving the Law of Total Expectation

$$\mathbf{E}[\boldsymbol{X}]$$

The definition of $\mathrm{E}[X]$.

Proving the Law of Total Expectation

$$\mathbf{E}[X]$$

Which is the left-hand side.

The Thing We Never Defined

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What is utility?

What Utility Is

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- **Utility as pleasure.** High utility to the extent that the outcome increases pleasure and reduces pain. MEU then says: maximise expected pleasure.
- **Utility as value.** Utility to the extent that the outcome is actually valuable — prudentially, morally, or in another sense. MEU becomes MEV: maximise expected value.

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What Utility Is, Continued

- **Utility as degree of desire.** How much the agent wants the outcome, all things considered.
- **Utility as a pure representative device.** No psychological connection with the agent: a mathematical representation of preferences she already has.
- Words can mean what we want them to. I use "utility" for the representative device. That is today's sense.

Misconception 1

BDRC

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- **False.** Psychological hedonism is a thesis about what *motivates* people.
- The MEP Principle is a thesis about what people *should* do: what they do insofar as they are rational.
- So the two are independent. Either could be true while the other is false.

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- Nothing requires utility to match *how much* the agent wants an outcome.

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- If utility represents preferences, then higher utility means preferred. That is the *only* sense in which representative utility measures what the agent wants.
- Nothing requires utility to match *how much* the agent wants an outcome.
- What utility tracks is the agent's attitude to **risk**: a much higher utility means she will accept even small probabilities of that outcome. That is all it means.

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- But that is not the only way. We might be required to care about certain things: love, friendship, pleasure, carrot cake. We might be required always to do the morally right thing. Rationality might require going off into the woods to commune with god.
- We assume neither. Our assumptions about rationality are deliberately minimal, and neutral between these visions.

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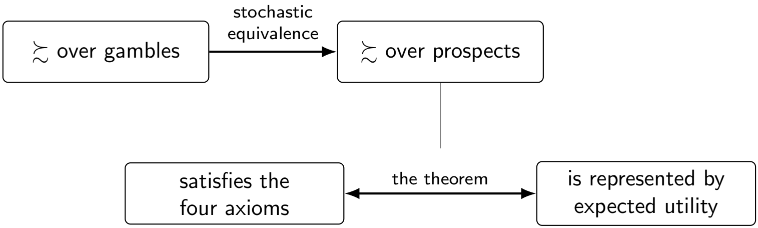
A representation theorem. The numbers are read off the preferences.

Plan


$\succsim$ over gambles
stochastic equivalence
$\succsim$ over prospects
satisfies the four axioms
the theorem
is represented by expected utility

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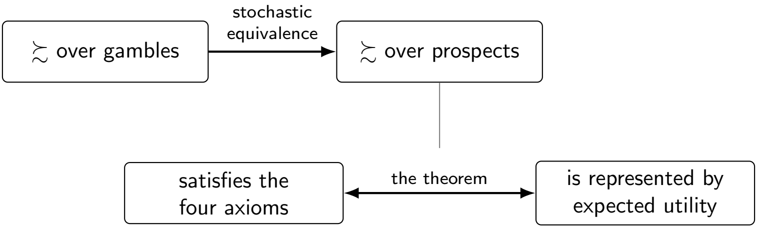

- Two steps. First get from gambles to prospects. Then show that the axioms hold of $\succsim$ over prospects **if and only if** $\succsim$ is represented by a utility function.

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- $\succsim$ is the basic notion. The other two are defined from it.

Preference and Choice

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- It is defined over every pair of gambles, not just the choices the agent actually faces.

What Gets Ranked

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- $\succsim$ ranges over **all** gambles — every act in every decision table, including ones nobody has offered.

The Ordering Principles

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- Both are among the assumptions of expected utility theory.

Some Facts About the Preference Order

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- Suppose $A \sim B$ and $B \succ C$. What about A and C? $A \succ C$.
- Can $A \succ B$ and $B \succ A$ both hold? No. $A \succ B$ says not $B \succsim A$.

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- Suppose $A \sim B$ and $B \succ C$. What about A and C?
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- Can $A \succ B$ and $B \succ A$ both hold? No. $A \succ B$ says not $B \succsim A$.

- Each follows from the definitions plus transitivity. We'll use facts like these without comment.

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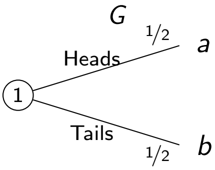

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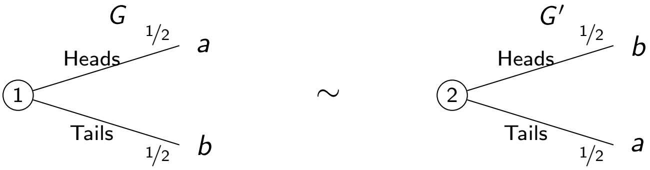

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- Only the **chances of the final outcomes** matter. Not which states carry them.

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- Caspar Hare, "Take the Sugar", *Analysis* 70 (2010).
- Tomi Francis and Johan Gustafsson, "Stochastic Dominance for Incomplete Preferences", *Analysis* 86 (2026), 299–311.

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Definition

The **prospect of a gamble** G, written P_G, is the function taking each outcome x to the probability that G results in x — computed from your credences in the states.

Same Prospect, Different Gambles

	Heads (1/2)	**Tails** (1/2)
G	a	b
G'	b	a

$$P_G(a) = 1/2, \quad P_G(b) = 1/2$$
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- Stochastic equivalence says exactly that $G \sim G'$.

From Gambles to Prospects

Proposition

Suppose $\succsim$ satisfies stochastic equivalence and transitivity. If $P_G = P_{G'}$ and $P_H = P_{H'}$, then $G \succsim H$ iff $G' \succsim H'$.

$$G' \sim G, \qquad H \sim H'$$

Stochastic equivalence: same prospect, so indifferent.

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Suppose $G \succsim H$.

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$$\boldsymbol{G' \succsim G \succsim H}$$

$G' \sim G$ gives $G' \succsim G$.

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$$\boldsymbol{G'} \succsim \boldsymbol{H} \succsim H'$$

Transitivity. And $H \sim H'$ gives $H \succsim H'$.

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Transitivity again. The other direction is the same argument with the primes swapped.

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$$\boldsymbol{G' \succsim H'}$$

Which gambles carry the prospects makes no difference.

Preferences Over Prospects

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- Assumed, not proved here: if the ordering properties hold of $\succsim$ over gambles then, given stochastic equivalence, they hold of $\succsim$ over prospects.
- Also in the background: every prospect is the prospect of *some* gamble.

Mixing Prospects

Definition

For prospects A and B and $p \in [0,1]$, $A\,p\,B$ is the prospect with

$$(A\,p\,B)(x) = p \cdot A(x) + (1-p) \cdot B(x) \qquad \text{for each outcome } x.$$

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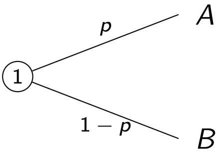
- "A with probability p, otherwise B."
- A and B need not be single outcomes. They can themselves be prospects.

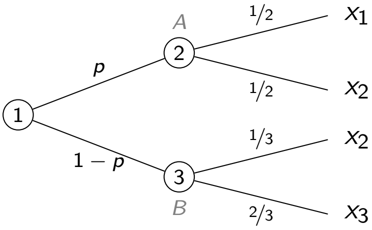
Mixing, in a Picture

$A\,p\,B$, where A and B are prospects.



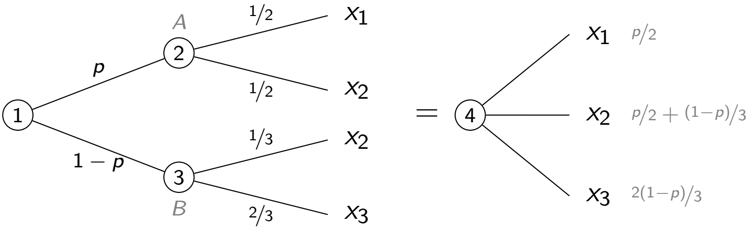
Mixing, in a Picture

Open up A and B: each is itself a spread of chances over outcomes.



Mixing, in a Picture

Multiply through, and it is one prospect over the outcomes.



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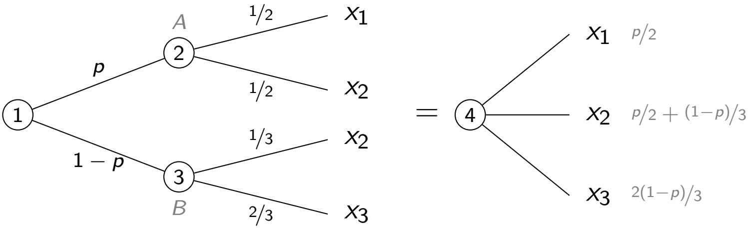

- This is an **equation**, not a preference. Prospects with the same chances *are* the same prospect.

The Mixture Space

- The set of all prospects over the outcomes, together with the mixing operation, is a **mixture space**.

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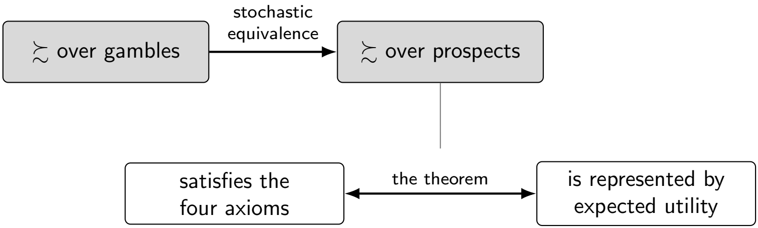
- From here on, $\succsim$ is a relation over prospects. This is justified by the Stochastic Equivalence assumption.
- Single outcomes can be identified with prospects too: x with the prospect that gives x for certain.

What We Are Going to Prove



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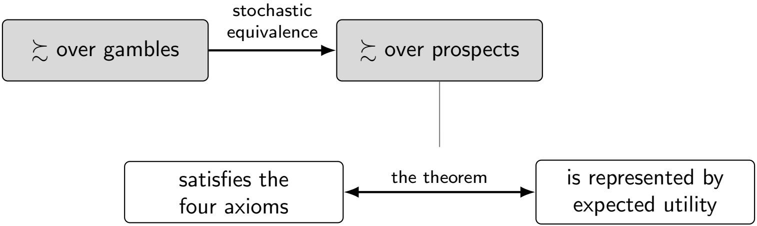

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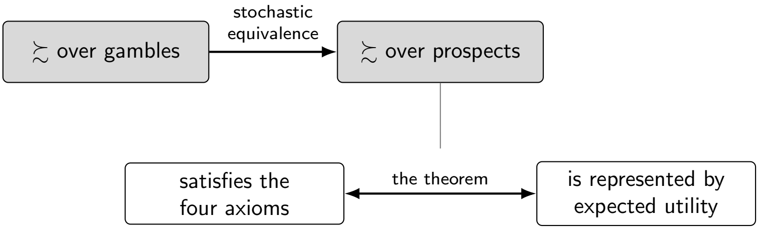

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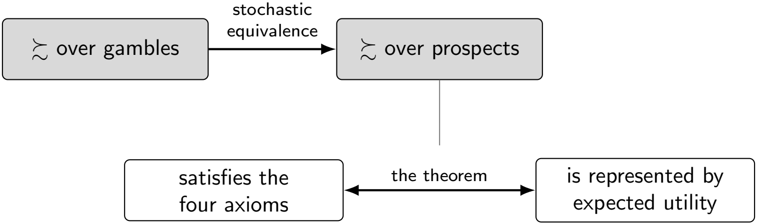

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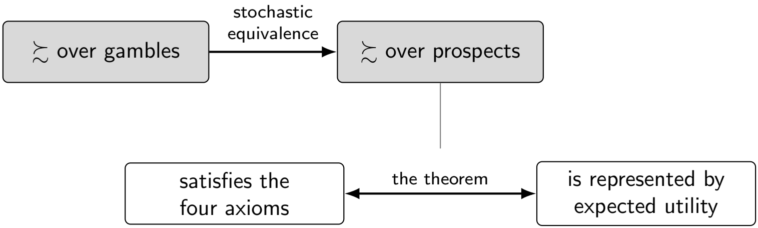

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- Two axioms so far: reflexivity and transitivity. Three to come.

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For all prospects A and B: $A \succsim B$ or $B \succsim A$.

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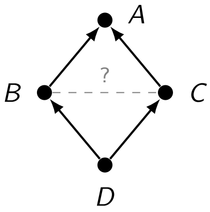

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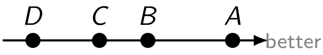

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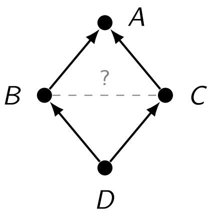

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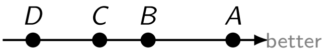

- No gaps: every pair is comparable. With transitivity, everything lines up.

Independence

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For all prospects A, B, C and every $p \in (0,1)$: $A \succsim B$ iff $A\,p\,C \succsim B\,p\,C$ iff $C\,p\,A \succsim C\,p\,B$.

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	E $(\mathrm{Cr}(E) = p)$	$\neg E$ $(1-p)$
$A\,p\,C$	A	C
$B\,p\,C$	B	C

- One way to realise them: gambles that agree if $\neg E$. But any gambles with these prospects will do; they need not agree anywhere.

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- One way to realise them: gambles that agree if $\neg E$. But any gambles with these prospects will do; they need not agree anywhere.
- Apply it twice: $A\,p\,C \succsim B\,p\,C$ iff $A \succsim B$ iff $A\,q\,C \succsim B\,q\,C$. **So the weight makes no difference.**

Independence and the Sure-Thing Principle

Sure-thing principle

Gambles G, G', H, H' and an event E. If G, H agree if E and so do G', H', while G, G' agree if $\neg E$ and so do H, H': then $G \succsim G'$ iff $H \succsim H'$.

Two gambles that agree on what happens if $\neg E$.

	$E \quad (p)$	$\neg E \quad (1-p)$
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The shared outcome, in bold.

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Replace it by another shared outcome, D.

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H	A	$\boldsymbol{D}$
H'	B	$\boldsymbol{D}$

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- A principle about *states*: what the two gambles share cannot affect the comparison.
- Independence implies it. With stochastic equivalence the two come to the same thing, at least when there are finitely many states.

Continuity

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If $A \succ B \succ C$, then there is some $p \in (0,1)$ with $B \sim A\,p\,C$.

B sits strictly between A and C.

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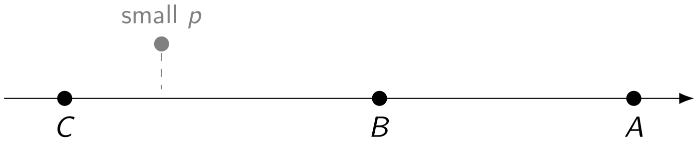
If $A \succ B \succ C$, then there is some $p \in (0,1)$ with $B \sim A\,p\,C$.

$A\,p\,C$ runs from C (at $p = 0$) to A (at $p = 1$).



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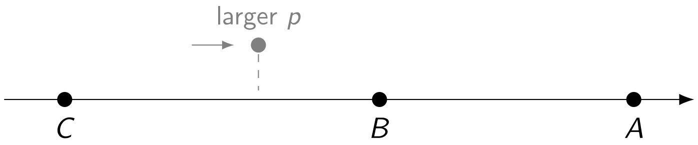
Somewhere on the way it must draw level with B.



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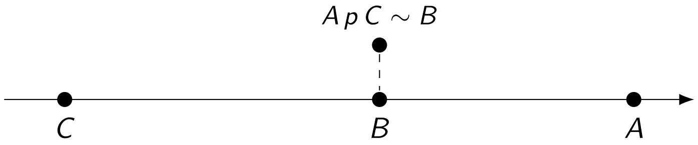
That p is what Continuity promises.



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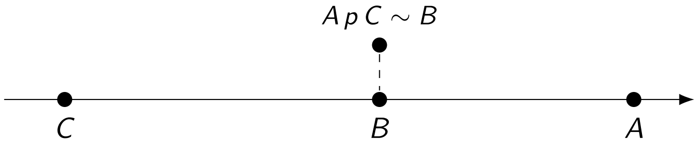

- No prospect is *infinitely* better than another.

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- The question is whether such preferences could be **rational**.

The Axioms

Reflexivity	$\boldsymbol{A \succsim A}$
Transitivity	$A \succsim B$ and $B \succsim C$ give $A \succsim C$
Completeness	$A \succsim B$ or $B \succsim A$
Independence	$A \succsim B$ iff $A\,p\,C \succsim B\,p\,C$ iff $C\,p\,A \succsim C\,p\,B$
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- Stochastic equivalence is assumed in the background.

The Expected Utility Theorem

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Let $\succsim$ be a preference relation over prospects.

(i) $\succsim$ satisfies the four axioms **if and only if** some function u on outcomes represents it by expected utility: for all prospects P and Q,

$$P \succsim Q \quad \text{iff} \quad \mathrm{EU}(P) \geq \mathrm{EU}(Q),$$

where $\mathrm{EU}(P) = \sum_x P(x)\, u(x)$, the sum running over the finitely many outcomes P gives positive probability to.

(ii) Any two such functions u and u' differ only in unit and zero: $u' = a \cdot u + c$ for some $a > 0$ and some c.

The Idea of the Proof

- Pick two prospects, one better than the other: $A_1 \succ A_0$. The choice is more or less arbitrary.

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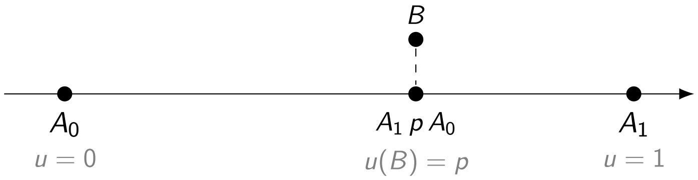
- Pick two prospects, one better than the other: $A_1 \succ A_0$. The choice is more or less arbitrary.
- Assign $u(A_1) = 1$ and $u(A_0) = 0$: the unit and the zero.
- For any B between them, there is exactly one p with $B \sim A_1\, p\, A_0$. Assign $u(B) = p$.



The Idea of the Proof, Continued

- If $B \succ A_1$: there is exactly one p with $A_1 \sim B\,p\,A_0$. Assign $u(B) = {}^1\!/_p$.

The Idea of the Proof, Continued

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- The other direction: preferences that violate any one of the axioms cannot be represented by a utility function.

Four Facts About Mixing

(M1) $A\,p\,A = A$.

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(M2) $A\,p\,B = B\,(1-p)\,A.$

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- All four are arithmetic. Compute the chance of each outcome on both sides.
- The proof leans on (M1) and (M4).

Beginning the Proof

- The proof follows Fishburn, *The Foundations of Expected Utility* (1982), ch. 2, Theorem 1. The same proof is Theorem 8.4 of his *Utility Theory for Decision Making* (1970).

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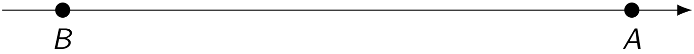
- If no prospect is strictly preferred to any other, set u constant. Done.
- Otherwise fix two prospects $A_1 \succ A_0$: the **unit** and the **zero**.

Step 1: More of the Better Thing Is Better

Lemma 1

If $A \succ B$ and $p, q \in [0,1]$, then $A\,p\,B \succ A\,q\,B$ iff $p > q$.



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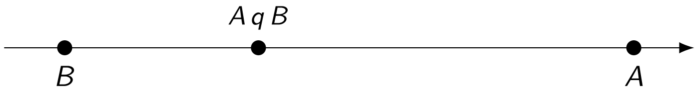

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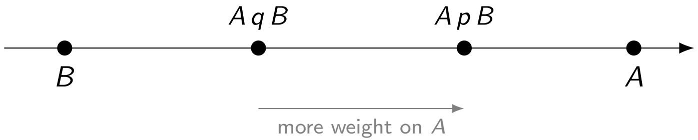

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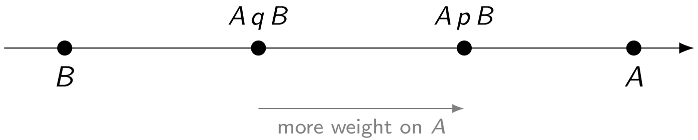

- Sliding weight from B onto A moves you up.

Proving Lemma 1

Given $A \succ B$ and $0 < q < p < 1$ (the edge cases $q = 0$ and $p = 1$ take one step each). Write $s = {}^{(p-q)}/_{(1-q)}$, so that $s + (1-s)q = p$.

$$A \succ B$$

The hypothesis.

Proving Lemma 1

Given $A \succ B$ and $0 < q < p < 1$ (the edge cases $q = 0$ and $p = 1$ take one step each). Write $s = {}^{(p-q)}/_{(1-q)}$, so that $s + (1-s)q = p$.

$$\boldsymbol{A\,q\,A \succ A\,q\,B}$$

Independence, mixing A in front of both sides at weight q.

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$$\boldsymbol{A} \succ A\,q\,B$$

(M1) on the left.

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$$\boldsymbol{A\, s\, (A\, q\, B) \succ (A\, q\, B)\, s\, (A\, q\, B)}$$

Independence again, mixing both sides with $A\, q\, B$ at weight s.

Proving Lemma 1

Given $A \succ B$ and $0 < q < p < 1$ (the edge cases $q = 0$ and $p = 1$ take one step each). Write $s = {}^{(p-q)}/_{(1-q)}$, so that $s + (1-s)q = p$.

$$A\, s\, (A\, q\, B) \succ \boldsymbol{A\, q\, B}$$

(M1) on the right.

Proving Lemma 1

Given $A \succ B$ and $0 < q < p < 1$ (the edge cases $q = 0$ and $p = 1$ take one step each). Write $s = {}^{(p-q)}/_{(1-q)}$, so that $s + (1 - s)q = p$.

$$\boldsymbol{A\,[s + (1 - s)q]\,B} \succ A\,q\,B$$

(M4) on the left, with s in the role of p.

Proving Lemma 1

Given $A \succ B$ and $0 < q < p < 1$ (the edge cases $q = 0$ and $p = 1$ take one step each). Write $s = {}^{(p-q)}/_{(1-q)}$, so that $s + (1-s)q = p$.

$$\boldsymbol{A\,p\,B} \succ A\,q\,B$$

$s + (1-s)q = p$, by the choice of s.

Proving Lemma 1

Given $A \succ B$ and $0 < q < p < 1$ (the edge cases $q = 0$ and $p = 1$ take one step each). Write $s = {}^{(p-q)}/_{(1-q)}$, so that $s + (1-s)q = p$.

$$\boldsymbol{A\,p\,B} \succ A\,q\,B$$

One direction of Lemma 1. For the other: if $p = q$ the two sides are the same prospect, and if $p < q$ the argument just given puts $A\,q\,B$ strictly above $A\,p\,B$. Either way, not $A\,p\,B \succ A\,q\,B$.

Step 2: A Mixture of Worse (or Equal) Prospects Is Worse (or Equal)

Lemma 2

If $A \succsim X$ and $A \succsim Y$, then $A \succsim X\, p\, Y$.

$$A \succsim X$$

The first hypothesis.

Step 2: A Mixture of Worse (or Equal) Prospects Is Worse (or Equal)

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If $A \succsim X$ and $A \succsim Y$, then $A \succsim X\,p\,Y$.

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If $A \succsim X$ and $A \succsim Y$, then $A \succsim X\,p\,Y$.

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$$\boldsymbol{A} \succsim A \, p \, Y$$

(M1) on the left.

Step 2: A Mixture of Worse (or Equal) Prospects Is Worse (or Equal)

Lemma 2

If $A \succsim X$ and $A \succsim Y$, then $A \succsim X\,p\,Y$.

$$A \succsim A\,p\,Y \quad \text{and} \quad A\,p\,Y \succsim X\,p\,Y$$

Put the two results together.

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If $A \succsim X$ and $A \succsim Y$, then $A \succsim X \, p \, Y$.

$$\boldsymbol{A \succsim X \, p \, Y}$$

Transitivity.

Step 2: A Mixture of Worse (or Equal) Prospects Is Worse (or Equal)

Lemma 2

If $A \succsim X$ and $A \succsim Y$, then $A \succsim X\,p\,Y$.

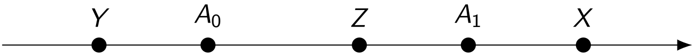
$$\boldsymbol{A \succsim X\,p\,Y}$$

A mixture of prospects that A is at least as good as is itself one that A is at least as good as. The same proof with the preferences reversed gives the mirror image.

Intervals

Definition

For prospects $A \succsim B$, the **interval** $[B, A]$ is the set of prospects X with $A \succsim X \succsim B$.



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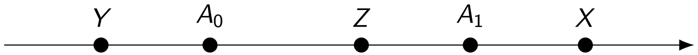

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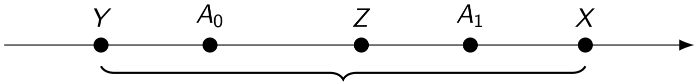

$[B, A]$, with B the worst of them and A the best

- Lemma 2: an interval is **closed under mixing**. If X and Y are in $[B, A]$, so is $X\,q\,Y$.
- Any **finitely many** prospects have a best and a worst among them, by Completeness and Transitivity. So they all lie in one interval.

Step 3: Calibration on an Interval

Lemma 3

Fix $A \succ B$. For every X in $[B, A]$ there is **exactly one** $p \in [0, 1]$ with $X \sim A\,p\,B$.

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- Otherwise $A \succ X \succ B$, and Continuity hands us a p.
- **Uniqueness.** Two different values $p > q$ would give $A\,p\,B \succ A\,q\,B$, by Lemma 1.

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Lemma 1, again

If $A \succ B$ and $p, q \in [0, 1]$, then $A\,p\,B \succ A\,q\,B$ iff $p > q$.

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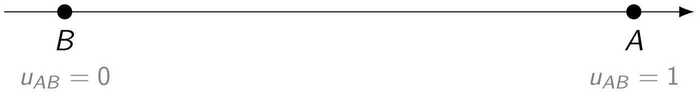
- Otherwise $A \succ X \succ B$, and Continuity hands us a p.
- **Uniqueness.** Two different values $p > q$ would give $A\,p\,B \succ A\,q\,B$, by Lemma 1.
- But both are $\sim X$. Contradiction, by transitivity.

Defining u_{AB}

Definition

For X in $[B, A]$: $u_{AB}(X)$ is the unique p with $X \sim A\,p\,B$.



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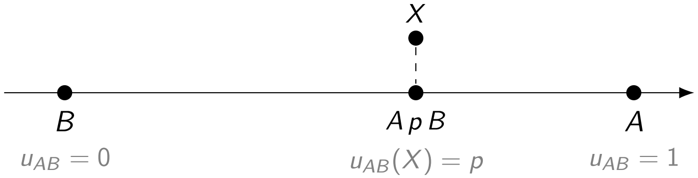

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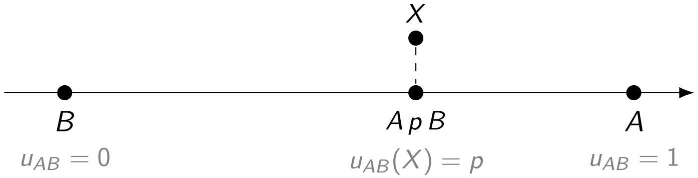

- One calibration per interval. The subscript keeps track of which ends were used.

Step 4: u_{AB} Represents $\succsim$ on the Interval

Lemma 4

For X and Y in $[B, A]$: $X \succsim Y$ iff $u_{AB}(X) \geq u_{AB}(Y)$.

Write $p = u_{AB}(X)$ and $q = u_{AB}(Y)$, so that $X \sim A\,p\,B$ and $Y \sim A\,q\,B$.

$$X \succsim Y$$

Start from the left-hand side.

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$$X \succsim Y \quad \text{iff} \quad \boldsymbol{A\,p\,B} \succsim \boldsymbol{A\,q\,B}$$

Swap X and Y for the prospects they are indifferent to. Transitivity, in both directions.

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$$X \succsim Y \quad \text{iff} \quad \text{not } \boldsymbol{A\,q\,B \succ A\,p\,B}$$

Completeness: $P \succsim Q$ holds exactly when $Q \succ P$ fails.

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$$X \succsim Y \quad \text{iff} \quad \text{not } \boldsymbol{q} > \boldsymbol{p}$$

Lemma 1, in both directions: $A\,q\,B \succ A\,p\,B$ iff $q > p$.

Step 4: u_{AB} Represents $\succsim$ on the Interval

Lemma 4

For X and Y in $[B, A]$: $X \succsim Y$ iff $u_{AB}(X) \geq u_{AB}(Y)$.

Write $p = u_{AB}(X)$ and $q = u_{AB}(Y)$, so that $X \sim A\,p\,B$ and $Y \sim A\,q\,B$.

$$\boldsymbol{X \succsim Y} \quad \text{iff} \quad \boldsymbol{p \geq q}$$

And $p \geq q$ is $u_{AB}(X) \geq u_{AB}(Y)$.

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$$\boldsymbol{X} \succsim \boldsymbol{Y} \quad \text{iff} \quad \boldsymbol{p} \geq \boldsymbol{q}$$

Which is Lemma 4. Every step was an iff.

Step 5: u_{AB} Respects Mixing on the Interval

Lemma 5

For X and Y in $[B, A]$ and $p \in [0, 1]$:
$u_{AB}(X\,p\,Y) = p \cdot u_{AB}(X) + (1 - p) \cdot u_{AB}(Y)$.

Write $q = u_{AB}(X)$ and $r = u_{AB}(Y)$, so that $X \sim A\,q\,B$ and $Y \sim A\,r\,B$.

$$X\,p\,Y$$

Start with the mixture. It is in $[B, A]$ by Lemma 2, so u_{AB} is defined on it.

Step 5: u_{AB} Respects Mixing on the Interval

Lemma 5

For X and Y in $[B, A]$ and $p \in [0,1]$:
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Write $q = u_{AB}(X)$ and $r = u_{AB}(Y)$, so that $X \sim A\,q\,B$ and $Y \sim A\,r\,B$.

$$X\,p\,Y \sim \boldsymbol{(A\,q\,B)\,p\,Y}$$

$X \sim A\,q\,B$. Independence, mixing Y behind both at weight p.

Step 5: u_{AB} Respects Mixing on the Interval

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$$X\,p\,Y \sim \boldsymbol{(A\,q\,B)\,p\,(A\,r\,B)}$$

$Y \sim A\,r\,B$. Independence, mixing $A\,q\,B$ in front of both at weight p.

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$$X\,p\,Y \sim \boldsymbol{A\,[pq + (1-p)r]\,B}$$

Arithmetic: a mixture of two A–B mixtures is an A–B mixture, with the weights averaged.

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Write $q = u_{AB}(X)$ and $r = u_{AB}(Y)$, so that $X \sim A\,q\,B$ and $Y \sim A\,r\,B$.

$$\boldsymbol{u_{AB}(X\,p\,Y) = pq + (1 - p)r}$$

Lemma 3: that is the unique calibrating value.

Step 5: u_{AB} Respects Mixing on the Interval

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For X and Y in $[B, A]$ and $p \in [0, 1]$:
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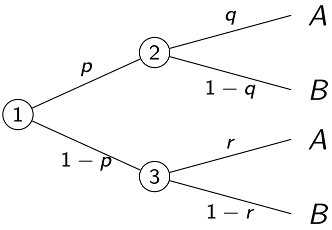
$$\boldsymbol{u_{AB}(X\,p\,Y) = pq + (1-p)r}$$

Which is Lemma 5.

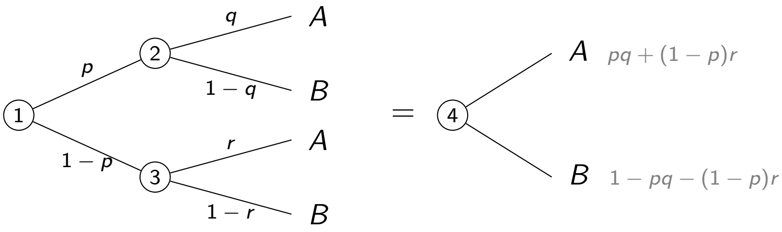
Lemma 5, in a Picture

$X\,p\,Y$, with X and Y replaced by their calibrations.



Lemma 5, in a Picture

Multiply through: A with total weight $pq + (1-p)r$, B with the rest.



Rescaling Keeps Both Properties

Suppose g represents $\succsim$ and respects mixing on some interval. Take $a > 0$ and any c, and put $h = a \cdot g + c$.

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- *h* **represents $\succsim$.** Since $a > 0$: $h(X) \geq h(Y)$ iff $a\,g(X) + c \geq a\,g(Y) + c$ iff $g(X) \geq g(Y)$ iff $X \succsim Y$.

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- h **respects mixing.** Using $c = p\,c + (1-p)\,c$:

$$\begin{aligned} h(X\,p\,Y) &= a\,[p\,g(X) + (1-p)\,g(Y)] + c \\ &= p\,[a\,g(X) + c] + (1-p)\,[a\,g(Y) + c] \\ &= p\,h(X) + (1-p)\,h(Y). \end{aligned}$$

Rescaling Keeps Both Properties

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- So a rescaling with a positive factor loses nothing. We use this twice: next, and for uniqueness.

The Five Lemmas

Lemma 1

If $A \succ B$ and $p, q \in [0, 1]$, then $A\,p\,B \succ A\,q\,B$ iff $p > q$.

Lemma 2

If $A \succsim X$ and $A \succsim Y$, then $A \succsim X\,p\,Y$; likewise with $\succsim$ reversed. So an interval is closed under mixing.

Lemma 3

Fix $A \succ B$. For every X in $[B, A]$ there is exactly one $p \in [0, 1]$ with $X \sim A\,p\,B$. That p is $u_{AB}(X)$.

Lemma 4

For X and Y in $[B, A]$: $X \succsim Y$ iff $u_{AB}(X) \geq u_{AB}(Y)$.

Lemma 5

For X and Y in $[B, A]$: $u_{AB}(X\,p\,Y) = p \cdot u_{AB}(X) + (1 - p) \cdot u_{AB}(Y)$.

Step 6: Beyond the Interval

- Take any prospect Z. Let A be the best of Z, A_1, A_0 and B the worst, so $[B, A]$ contains all three.

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- Calibrate on that interval, then **rescale** so that A_1 scores 1 and A_0 scores 0:

Definition

$$f(X) = \frac{u_{AB}(X) - u_{AB}(A_0)}{u_{AB}(A_1) - u_{AB}(A_0)} \qquad \text{for } X \text{ in } [B, A].$$

Step 6: Beyond the Interval

- Take any prospect Z. Let A be the best of Z, A_1, A_0 and B the worst, so $[B, A]$ contains all three.
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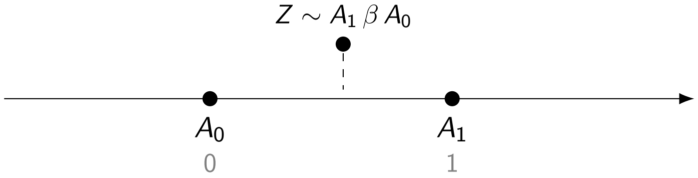
- Now three cases show what $f(Z)$ must be — **without reference to the interval**.

The Three Cases

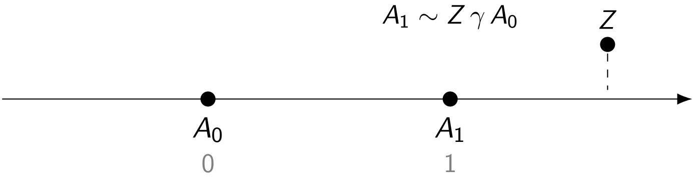
Case 1: $A_1 \succsim Z \succsim A_0$. Lemma 3: $Z \sim A_1 \, \beta \, A_0$, for exactly one β.



- Case 1: $f(Z) = \beta \cdot 1 + (1 - \beta) \cdot 0 = \beta$.

The Three Cases

Case 2: $Z \succ A_1$. Lemma 3 on $[A_0, Z]$: $A_1 \sim Z\,\gamma\,A_0$, for exactly one γ.



- Case 1: $f(Z) = \beta \cdot 1 + (1-\beta) \cdot 0 = \beta$.
- Case 2: $1 = f(A_1) = \gamma\, f(Z) + (1-\gamma) \cdot 0$, so $f(Z) = {}^1\!/\!\gamma$. And $\gamma > 0$, since $A_1 \succ A_0$.

The Three Cases

Case 3: $A_0 \succ Z$. Lemma 3 on $[Z, A_1]$: $A_0 \sim A_1 \, \alpha \, Z$, for exactly one α.

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- Case 3: $0 = f(A_0) = \alpha \cdot 1 + (1 - \alpha) \, f(Z)$, so $f(Z) = -{}^\alpha/_{(1-\alpha)}$. And $\alpha < 1$, likewise.

Why the Interval Doesn't Matter

- In each case, $f(Z)$ is fixed by an indifference that mentions only Z, A_1 and A_0. The interval played no part.

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This is exactly the assignment from the Idea of the Proof: β in between, $^1/_\gamma$ above, $-^{\alpha}/_{(1-\alpha)}$ below.

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- So $X \succsim Y$ iff $u(X) \geq u(Y)$, and $u(X\,q\,Y) = q\,u(X) + (1-q)\,u(Y)$.

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$$u(X) = p_1\, u(x_1) + p_2\, u(x_2) + \cdots + p_n\, u(x_n) = \mathrm{EU}(X).$$

- Since u represents $\succsim$: $X \succsim Y$ iff $\mathrm{EU}(X) \geq \mathrm{EU}(Y)$. **The representation.**

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- **Which** a **and** c**?** Only one choice is possible. At A_0: $v(A_0) = a \cdot u(A_0) + c = c$. At A_1: $v(A_1) = a \cdot u(A_1) + c = a + c$. So $c = v(A_0)$ and $a = v(A_1) - v(A_0)$.

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- $a > 0$, since v represents $\succsim$ and $A_1 \succ A_0$. So w is a positive rescaling of u: it represents $\succsim$ and respects mixing.
- **What is left:** show $v(Z) = w(Z)$ for every Z. Suppose not, for some Z. Three cases, each a contradiction.

Uniqueness: The Three Cases

v and w both represent $\succsim$ (so indifferent prospects get equal values), both respect mixing, and they agree at A_1 and A_0.

- **Case 1:** $A_1 \succsim Z \succsim A_0$, so $Z \sim A_1 \, \beta \, A_0$.

Both respect mixing, and they agree at the two ends:

$$v(Z) = \beta\, v(A_1) + (1 - \beta)\, v(A_0) = \beta\, w(A_1) + (1 - \beta)\, w(A_0) = w(Z).$$

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- **Case 1:** $A_1 \succsim Z \succsim A_0$, so $Z \sim A_1 \, \beta \, A_0$.
- **Case 2:** $Z \succ A_1$, so $A_1 \sim Z \, \gamma \, A_0$, with $\gamma > 0$.

Read the indifference through each function, then compare:

$$\gamma \, v(Z) + (1-\gamma) \, v(A_0) = v(A_1) = w(A_1) = \gamma \, w(Z) + (1-\gamma) \, w(A_0).$$

Cancel and divide by $\gamma > 0$: $v(Z) = w(Z)$.

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- **Case 3:** $A_0 \succ Z$, so $A_0 \sim A_1 \, \alpha \, Z$, with $\alpha < 1$.

The same, at the other end:

$$\alpha \, v(A_1) + (1-\alpha) \, v(Z) = v(A_0) = w(A_0) = \alpha \, w(A_1) + (1-\alpha) \, w(Z).$$

Cancel and divide by $1-\alpha > 0$: $v(Z) = w(Z)$.

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Contradiction in every case. So $v = w = a \cdot u + c$, with $a = v(A_1) - v(A_0) > 0$ and $c = v(A_0)$.

Rescaled Expected Utility Is Expected Utility

Take any $a > 0$ and c, and put $u' = a \cdot u + c$.

$$\mathrm{EU}'(P) = \sum_x P(x)\, \boldsymbol{u'(x)}$$

Expected utility computed with u'.

Rescaled Expected Utility Is Expected Utility

Take any $a > 0$ and c, and put $u' = a \cdot u + c$.

$$\mathrm{EU}'(P) = \sum_x P(x) \, \boldsymbol{[a \, u(x) + c]}$$

Substitute $u' = a \, u + c$.

Rescaled Expected Utility Is Expected Utility

Take any $a > 0$ and c, and put $u' = a \cdot u + c$.

$$\mathrm{EU}'(P) = \boldsymbol{a} \sum_{\boldsymbol{x}} \boldsymbol{P(x)\,u(x)} + \boldsymbol{c} \sum_{\boldsymbol{x}} \boldsymbol{P(x)}$$

Collect the a terms and the c terms.

Rescaled Expected Utility Is Expected Utility

Take any $a > 0$ and c, and put $u' = a \cdot u + c$.

$$\mathrm{EU}'(P) = a \sum_x P(x)\, u(x) + c \cdot \mathbf{1}$$

The probabilities sum to 1.

Rescaled Expected Utility Is Expected Utility

Take any $a > 0$ and c, and put $u' = a \cdot u + c$.

$$\mathbf{EU'(\boldsymbol{P}) = \boldsymbol{a} \cdot EU(\boldsymbol{P}) + \boldsymbol{c}}$$

An increasing linear function of EU.

Rescaled Expected Utility Is Expected Utility

Take any $a > 0$ and c, and put $u' = a \cdot u + c$.

$$\mathbf{EU'(P) = a \cdot EU(P) + c}$$

Since $a > 0$: $\mathrm{EU}'(P) \geq \mathrm{EU}'(Q)$ exactly when $\mathrm{EU}(P) \geq \mathrm{EU}(Q)$.

Step 9: The Other Direction

Suppose $\succsim$ is represented by expected utility for some u. Then each axiom holds.

- **Transitivity and Completeness.**

$\geq$ on numbers is transitive and complete, and $\succsim$ copies it: $P \succsim Q$ iff $\mathrm{EU}(P) \geq \mathrm{EU}(Q)$.

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Suppose $\succsim$ is represented by expected utility for some u. Then each axiom holds.

- **Transitivity and Completeness.**
- **Independence.**

$$\mathrm{EU}(A\,p\,C) = p\,\mathrm{EU}(A) + (1-p)\,\mathrm{EU}(C).$$ So

$$\mathrm{EU}(A\,p\,C) \geq \mathrm{EU}(B\,p\,C) \text{ iff } p\,\mathrm{EU}(A) \geq p\,\mathrm{EU}(B) \text{ iff } \mathrm{EU}(A) \geq \mathrm{EU}(B),$$

since $p > 0$.

The same for $C\,p\,A$ and $C\,p\,B$, since $1 - p > 0$.

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- **Transitivity and Completeness.**
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- **Continuity.**

Given $\mathrm{EU}(A) > \mathrm{EU}(B) > \mathrm{EU}(C)$, put $p = {}^{[\mathrm{EU}(B)-\mathrm{EU}(C)]}/_{[\mathrm{EU}(A)-\mathrm{EU}(C)]}$, which lies in $(0, 1)$. Then

$$\mathrm{EU}(A\,p\,C) = \mathrm{EU}(C) + p\,[\mathrm{EU}(A) - \mathrm{EU}(C)] = \mathrm{EU}(B),$$

so $B \sim A\,p\,C$.

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Both directions of (i), and (ii). The theorem is proved.

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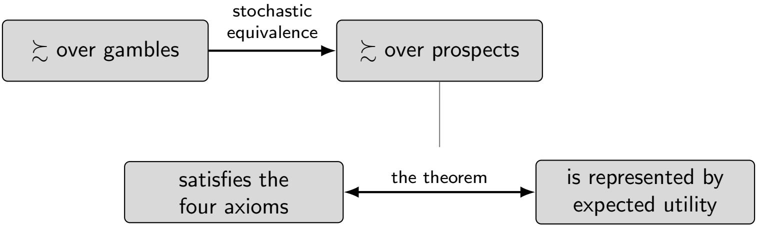

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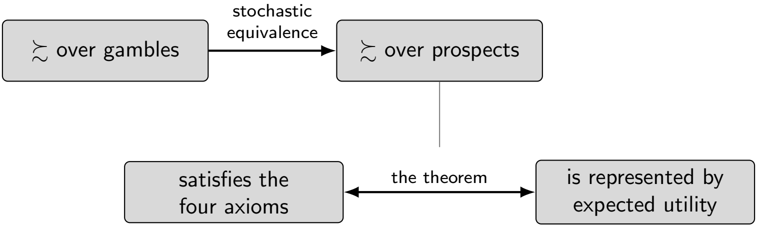

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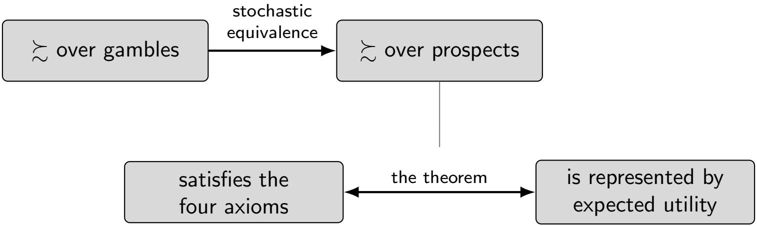

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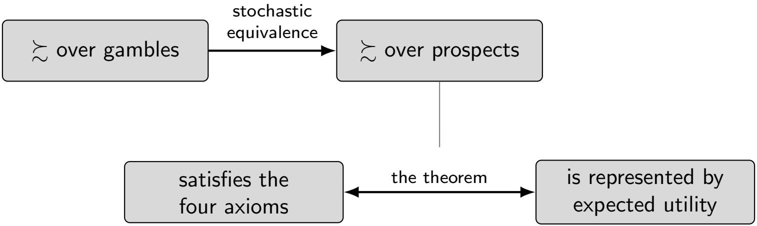

- Four axioms on preference hold exactly when some utility function on outcomes represents it — however many outcomes there are.
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- The function is fixed once the unit and zero are: an interval scale.

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- The theorem says the preferences *can be described* as expected-utility maximising.
- It does not say the agent computes anything. It does not say u is a quantity in her head that *causes* the preferences.

What Rationality Requires, on This View

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- Rather: have preferences satisfying the **axioms**.
- Maximising expected utility is then a *consequence*, not an instruction.

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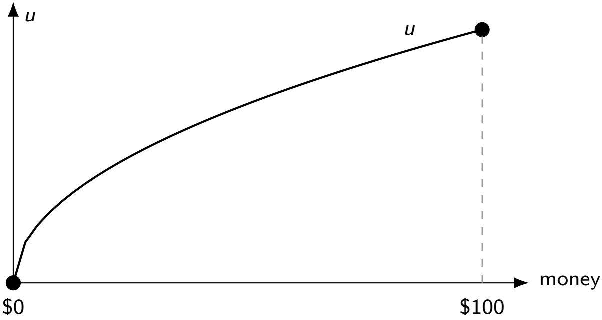
- Every $a \cdot u + c$ with $a > 0$ does the same job. Nothing else does.
- Ratios of *differences* do mean something: they survive every rescaling.

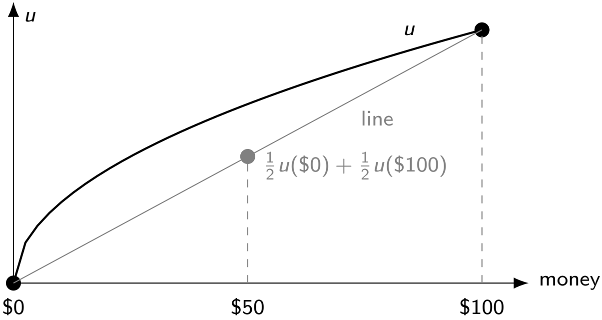
Risk Aversion: Two Senses

The black curve is u, the utility of money. A fair coin for \$100 or \$0, against \$50 for sure.



Risk Aversion: Two Senses

The grey line joins $u(\$0)$ to $u(\$100)$. Its midpoint is the gamble's expected utility.



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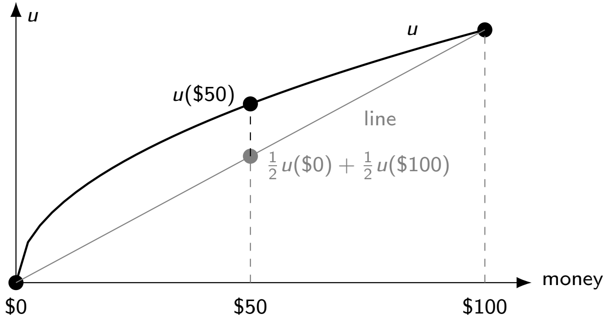
A concave u puts $u(\$50)$ **above** the line: the sure thing wins. Risk aversion about money, inside EUT.



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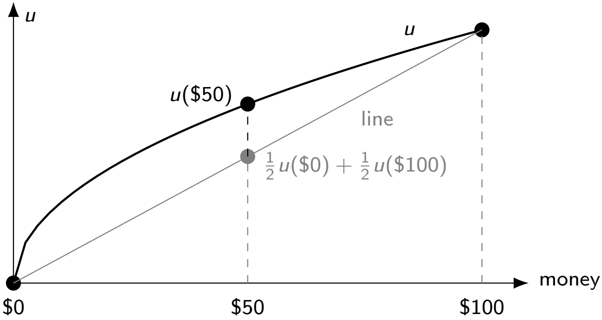

- Risk aversion about **utility itself** — preferring a sure thing to a gamble with the same *expected utility* — is ruled out.

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The Axioms, Again

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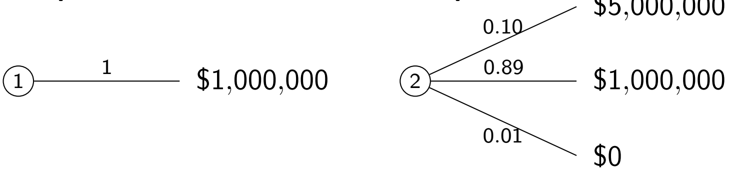
- **Independence.** Now: the Allais paradox.

A Choice

Which would you take?

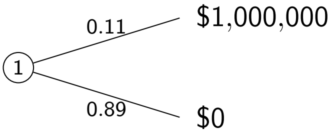
Option 1 **Option 2**



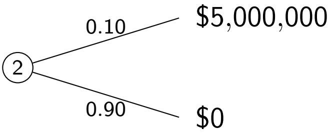
Another Choice

And now?

Option 3



Option 4



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Write Z for: \$5,000,000 with probability $^{10}/_{11}$, else \$0.

	with probability 0.11	with probability 0.89
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Option 2	Z	\$1,000,000
Option 3	\$1,000,000	\$0
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Option 4	Z	\$0

- Within each pair, the second column is **shared**.

What Independence Says About It

Write M for \$1,000,000 for certain, and O for \$0 for certain.

$$\text{Option 1} = M\,0.11\,M \qquad \text{Option 3} = M\,0.11\,O$$
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- Independence: Option 1 $\succsim$ Option 2 iff $M \succsim Z$ iff Option 3 $\succsim$ Option 4.

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- Independence: Option 1 $\succsim$ Option 2 iff $M \succsim Z$ iff Option 3 $\succsim$ Option 4.
- So Option 1 $\succ$ Option 2 together with Option 4 $\succ$ Option 3 **violates Independence**.

Ellsberg's Urn

30 red balls, and 60 that are black or yellow in an unknown proportion.
One ball is drawn.

	red (30)	black (?)	yellow (?)
Bet I	\$100	\$0	\$0
Bet II	\$0	\$100	\$0
Bet III	\$100	\$0	\$100
Bet IV	\$0	\$100	\$100

Ellsberg's Urn

30 red balls, and 60 that are black or yellow in an unknown proportion.
One ball is drawn.

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- No probabilistic credence in "black" makes both preferences expected-utility maximising.

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- **Give up Independence.** Theories that weight probabilities by rank, or by the agent's attitude to risk.

Next Week

Money pumps and dynamic choice

- Gustafsson, *Money-Pump Arguments*, chs. 1–2 — on the website
- BDRC ch. 8 — optional