PE3101P: Decision and Social Choice

Optional Exercises (Week 4) — Solutions

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1. Let Cr be an agent's credence function satisfying the probability axioms on a sample space Ω, and let B be an event with $\mathrm{Cr}(B) > 0$. Define

$$\mathrm{Cr}_B(A) = \mathrm{Cr}(A \mid B) = \frac{\mathrm{Cr}(A \cap B)}{\mathrm{Cr}(B)}.$$

Prove that Cr_B satisfies each probability axiom:

(a) Non-negativity: $\mathrm{Cr}_B(A) \geq 0$ for every event A.

(b) Normalisation: $\mathrm{Cr}_B(\Omega) = 1$.

(c) Additivity: if $X \cap Y = \varnothing$, then $\mathrm{Cr}_B(X \cup Y) = \mathrm{Cr}_B(X) + \mathrm{Cr}_B(Y)$.

You may use basic facts about sets without proving them.

Solution.

(a) $\mathrm{Cr}(A \cap B) \geq 0$ by Non-negativity, and $\mathrm{Cr}(B) > 0$. Therefore

$$\mathrm{Cr}_B(A) = \frac{\mathrm{Cr}(A \cap B)}{\mathrm{Cr}(B)} \geq 0.$$

(b) Since $\Omega \cap B = B$,

$$\mathrm{Cr}_B(\Omega) = \frac{\mathrm{Cr}(\Omega \cap B)}{\mathrm{Cr}(B)} = \frac{\mathrm{Cr}(B)}{\mathrm{Cr}(B)} = 1.$$

(c) If $X \cap Y = \varnothing$, then $(X \cap B) \cap (Y \cap B) = \varnothing$. Hence

$$\begin{aligned}
\mathrm{Cr}_B(X \cup Y) &= \frac{\mathrm{Cr}((X \cup Y) \cap B)}{\mathrm{Cr}(B)} \\
&= \frac{\mathrm{Cr}((X \cap B) \cup (Y \cap B))}{\mathrm{Cr}(B)} \\
&= \frac{\mathrm{Cr}(X \cap B) + \mathrm{Cr}(Y \cap B)}{\mathrm{Cr}(B)} \\
&= \mathrm{Cr}_B(X) + \mathrm{Cr}_B(Y).
\end{aligned}$$

The third equality uses Additivity for the two disjoint intersections.

2. An agent assigns the following credences to the elementary outcomes in $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. A random variable X takes the values shown:

	ω_1	ω_2	ω_3	ω_4
Credence	1/8	3/8	1/4	1/4
X	-4	0	4	8

Let $B = \{\omega_1, \omega_2\}$. Expectations are calculated using the agent's credences. For a random variable with possible values $x_1, \ldots, x_n$, recall

$$\begin{aligned}
\mathrm{E}[X] &= x_1 \,\mathrm{Cr}(X = x_1) + \cdots + x_n \,\mathrm{Cr}(X = x_n) \\
&= \sum_{j=1}^{n} x_j \,\mathrm{Cr}(X = x_j), \\
\mathrm{E}[X \mid B] &= x_1 \,\mathrm{Cr}(X = x_1 \mid B) + \cdots + x_n \,\mathrm{Cr}(X = x_n \mid B) \\
&= \sum_{j=1}^{n} x_j \,\mathrm{Cr}(X = x_j \mid B).
\end{aligned}$$

Calculate the following, showing your working:

(a) $\mathrm{E}[X]$.

(b) $\mathrm{E}[X \mid B]$.

(c) $\mathrm{E}[X \mid \neg B]$.

Solution.

(a) $\mathrm{E}[X] = (-4)\frac{1}{8} + 0\frac{3}{8} + 4\frac{1}{4} + 8\frac{1}{4} = \frac{5}{2}$.

(b) $\mathrm{Cr}(B) = \mathrm{Cr}(\{\omega_1\}) + \mathrm{Cr}(\{\omega_2\}) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$. Since both singleton events are subsets of B,

$$\begin{aligned}
\mathrm{Cr}(\{\omega_1\} \mid B) &= \frac{\mathrm{Cr}(\{\omega_1\} \cap B)}{\mathrm{Cr}(B)} = \frac{1/8}{1/2} = \frac{1}{8} \times 2 = \frac{1}{4}, \\
\mathrm{Cr}(\{\omega_2\} \mid B) &= \frac{\mathrm{Cr}(\{\omega_2\} \cap B)}{\mathrm{Cr}(B)} = \frac{3/8}{1/2} = \frac{3}{8} \times 2 = \frac{3}{4}.
\end{aligned}$$

The other two singleton events are disjoint from B, so their conditional credences are zero. Hence

$$\mathrm{E}[X \mid B] = (-4)\frac{1}{4} + 0\frac{3}{4} + 4(0) + 8(0) = -1.$$

(c) $\mathrm{Cr}(\neg B) = \mathrm{Cr}(\{\omega_3\}) + \mathrm{Cr}(\{\omega_4\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Since both singleton events are subsets of $\neg B$,

$$\begin{aligned}
\mathrm{Cr}(\{\omega_3\} \mid \neg B) &= \frac{\mathrm{Cr}(\{\omega_3\} \cap \neg B)}{\mathrm{Cr}(\neg B)} = \frac{1/4}{1/2} = \frac{1}{4} \times 2 = \frac{1}{2}, \\
\mathrm{Cr}(\{\omega_4\} \mid \neg B) &= \frac{\mathrm{Cr}(\{\omega_4\} \cap \neg B)}{\mathrm{Cr}(\neg B)} = \frac{1/4}{1/2} = \frac{1}{4} \times 2 = \frac{1}{2}.
\end{aligned}$$

The first two singleton events are disjoint from $\neg B$, so their conditional credences are zero. Hence

$$\mathrm{E}[X \mid \neg B] = (-4)(0) + 0(0) + 4\frac{1}{2} + 8\frac{1}{2} = 6.$$

3. For prospects A, B and $p \in [0,1]$, $A\ [p]\ B$ gives A with probability p and B with probability $1-p$. For each final outcome x, its probability is $pA(x) + (1-p)B(x)$.

The letters a, b, c denote distinct final outcomes, identified with prospects giving those outcomes for certain. Consider

$$P = (a\ [0.4]\ b)\ [0.25]\ (a\ [0.2]\ c), \qquad Q = a\ [0.25]\ (b\ [0.2]\ c).$$

(a) Calculate the probability of each final outcome under P.

(b) Calculate the probability of each final outcome under Q.

(c) Calculate the probability of outcome a under $P\ [1/12]\ Q$. You are not required to show your working.

Solution.

(a) $P(a) = 0.25(0.4) + 0.75(0.2) = 0.25$, $P(b) = 0.25(0.6) = 0.15$, and $P(c) = 0.75(0.8) = 0.60$.

(b) $Q(a) = 0.25$, $Q(b) = 0.75(0.2) = 0.15$, and $Q(c) = 0.75(0.8) = 0.60$.

(c) 0.25. Both P and Q assign probability 0.25 to a, so their mixture does too.

4. There are three final outcomes a, b, c, with scores $2, 1, 0$ respectively. For any prospect P, let $s(P)$ be the highest score among the outcomes to which P assigns positive probability. Define preferences by

$$P \succsim Q \quad \text{if and only if} \quad s(P) \geq s(Q).$$

Strict preference and indifference are defined from $\succsim$ in the usual way. In particular, equal scores give indifference.

Recall Continuity: if $X \succ Y \succ Z$, there is a $p \in (0,1)$ with $Y \sim X\ [p]\ Z$, where $X\ [p]\ Z$ gives X with probability p and Z otherwise.

Find prospects A, B, C such that $A \succ B \succ C$, and explain why no probability $p \in (0,1)$ gives $B \sim A\ [p]\ C$. Hence show that the stated preference rule violates Continuity.

Solution. Take $A = a$, $B = b$ and $C = c$, regarded as certain prospects. Their scores are $2, 1, 0$, so $A \succ B \succ C$. For every $p \in (0,1)$, $A\ [p]\ C$ gives positive probability to a, hence has score 2. Therefore $A\ [p]\ C \succ B$ for every such p. There is no p which makes the agent indifferent between $A[p]C$ and B, so Continuity fails.

5. An agent's preferences are represented by expected utility. The outcomes a, b, c, d, e have utilities $u(a) = 0$, $u(b) = 1$, with the remaining three values unknown. Write $A \sim B$ for indifference. A mixture $A\ [p]\ B$ gives A with probability p and B otherwise.

The agent has the following preferences:

$$c \sim b\ [0.3]\ a, \qquad b \sim d\ [0.25]\ a, \qquad a \sim b\ [0.5]\ e.$$

(a) Find $u(c)$, $u(d)$ and $u(e)$, showing how each indifference determines its value.

(b) Does the agent strictly prefer $d\ [0.2]\ e$ to a, strictly prefer a to that mixture, or regard them as equally preferred? Show your calculation.

Solution.

(a) Indifference gives equality of expected utilities. From $c \sim b\ [0.3]\ a$,

$$\begin{aligned} u(c) &= 0.3u(b) + 0.7u(a) \\ &= 0.3(1) + 0.7(0) \\ &= 0.3. \end{aligned}$$

From $b \sim d\ [0.25]\ a$,

$$\begin{aligned} u(b) &= 0.25u(d) + 0.75u(a), \\ 1 &= 0.25u(d) + 0.75(0), \\ 1 &= 0.25u(d), \\ u(d) &= \frac{1}{0.25} = 4. \end{aligned}$$

From $a \sim b\ [0.5]\ e$,

$$\begin{aligned} u(a) &= 0.5u(b) + 0.5u(e), \\ 0 &= 0.5(1) + 0.5u(e), \\ 0 &= 0.5 + 0.5u(e), \\ -0.5 &= 0.5u(e), \\ u(e) &= \frac{-0.5}{0.5} = -1. \end{aligned}$$

(b) $\mathrm{EU}(d\ [0.2]\ e) = 0.2(4) + 0.8(-1) = 0 = u(a)$. They are equally preferred.

6. An agent's preferences are represented by expected utility. Consider three outcomes a, b, c, with $u(a) = 1$ and $u(b) = 3$.

Write $A \succ B$ for strict preference. The mixture $c\ [p]\ a$ gives c with probability p and a otherwise. The agent's preferences must satisfy both

$$b \succ c\ [0.25]\ a \qquad \text{and} \qquad c\ [0.75]\ a \succ b.$$

(a) Find the full range of values of $u(c)$ for which both requirements hold. You may give your answer as an interval or as an inequality or inequalities.

(b) Give one value in this range and calculate the expected utilities of the two mixtures to test that it works.

(c) Now consider three new outcomes a', b', c', with $u(a') = 3$ and $u(b') = 5$. Find the full range of values of $u(c')$ for which

$$b' \succ c'\ [0.25]\ a' \qquad \text{and} \qquad c'\ [0.75]\ a' \succ b'.$$

You may give your answer as an interval or as an inequality or inequalities. You are not required to show your working.

Solution.

(a) Since preferences are represented by expected utility, $b \succ c\ [0.25]\ a$ requires

$$\begin{aligned} u(b) &> \tfrac{1}{4}u(c) + \tfrac{3}{4}u(a), \\ 3 &> \tfrac{1}{4}u(c) + \tfrac{3}{4}(1), \\ 3 - \tfrac{3}{4} &> \tfrac{1}{4}u(c), \\ \tfrac{9}{4} &> \tfrac{1}{4}u(c), \\ 9 &> u(c). \end{aligned}$$

Similarly, $c\ [0.75]\ a \succ b$ requires

$$\begin{aligned} \tfrac{3}{4}u(c) + \tfrac{1}{4}u(a) &> u(b), \\ \tfrac{3}{4}u(c) + \tfrac{1}{4}(1) &> 3, \\ \tfrac{3}{4}u(c) &> 3 - \tfrac{1}{4} = \tfrac{11}{4}, \\ u(c) &> \frac{11/4}{3/4} = \frac{11}{3}. \end{aligned}$$

Both requirements therefore hold exactly when $\frac{11}{3} < u(c) < 9$. The endpoints give indifference in one comparison and are excluded.

(b) Take $u(c) = 5$. The expected utilities are $\frac{1}{4}(5) + \frac{3}{4}(1) = 2$ and $\frac{3}{4}(5) + \frac{1}{4}(1) = 4$. Thus $u(b) = 3 > 2$ and $4 > 3 = u(b)$.

(c) The new fixed utilities are the old ones translated by 2: $u(a') = u(a) + 2$ and $u(b') = u(b) + 2$. Adding 2 to every outcome utility adds 2 to the expected utility of each mixture, so it preserves the same inequalities.

Thus the admissible values of $u(c')$ are exactly the old admissible values translated by 2:

$$\frac{11}{3} + 2 < u(c') < 9 + 2, \qquad \text{that is,} \qquad \frac{17}{3} < u(c') < 11.$$

Equivalently, subtracting 2 from all three new utilities recovers the inequalities in part (a), with $u(c') - 2$ in place of $u(c)$.

7. The final outcomes are monetary amounts $m \geq 0$. An agent has utility $u(m) = \sqrt{m}$ and ranks prospects by expected utility. Let L give \$100 with probability 1/2 and \$0 with probability 1/2.

(a) Calculate the expected monetary value and expected utility of L.

(b) Compare L with receiving \$36 for certain. Which does the agent prefer? Explain why this preference can differ from the ranking by expected monetary value.

(c) Find the monetary amount c such that the agent is indifferent between receiving c for certain and receiving L.

(d) Give a utility function $v(m)$ that increases with money and makes an agent strictly prefer L to certainty of \$50 if her preferences are representable by v. Show that your function gives this preference.

The function should be continuous. (You do not need to know what this means to answer the question. If you are unsure, it is highly unlikely that you will accidentally pick a function which is discontinuous.)

(e) Draw quick, rough sketches of u and your v, with money on the horizontal axis and utility on the vertical axis. What do you notice about the shapes of the two curves?

(f) Using these examples, suggest a relationship between the shape of a utility-of-money curve and an agent's attitude to monetary risk.

Solution.

(a) Expected money is $\frac{1}{2}(100) + \frac{1}{2}(0) = \$50$; expected utility is $\frac{1}{2}\sqrt{100} + \frac{1}{2}\sqrt{0} = 5$.

(b) Receiving \$36 for certain has utility $\sqrt{36} = 6 > 5$, so is preferred to L, even though its monetary value is less than L's expected monetary value of \$50.

The utilities are obtained by a nonlinear transformation of the monetary amounts. Consequently, the average of the utilities need not equal the utility of the average amount. Here,

$$\tfrac{1}{2}u(100) + \tfrac{1}{2}u(0) = 5, \qquad u\big(\tfrac{1}{2}(100) + \tfrac{1}{2}(0)\big) = \sqrt{50} \neq 5.$$

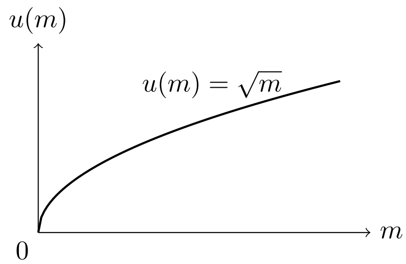
So comparing expected monetary amounts need not give the same ranking as comparing expected utilities.

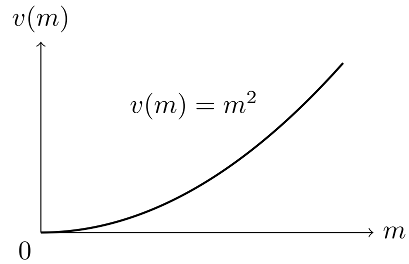
(c) $\sqrt{c} = 5$, so $c = \$25$.

(d) Take $v(m) = m^2$. It is continuous and strictly increasing for $m \geq 0$. Then

$$\mathrm{EU}_v(L) = \tfrac{1}{2}(100^2) + \tfrac{1}{2}(0^2) = 5000 > 2500 = v(50).$$

(e) The two sketches have different vertical scales.





$\sqrt{m}$ increases but flattens; m^2 increases and becomes steeper.

(f) Under expected-utility evaluation, concave utility of money corresponds to risk aversion: the sure expected monetary amount is at least as preferred as the lottery. Convex utility reverses this comparison; increasing linear utility gives indifference. Our two curves illustrate concavity and convexity respectively.

8. The outcomes a, b, c are distinct. Four prospects assign them the following probabilities:

	a	b	c
O_1	0	1	0
O_2	0.2	0.7	0.1
O_3	0	0.3	0.7
O_4	0.2	0	0.8

An agent strictly prefers O_1 to O_2 and O_4 to O_3.

Let D give a with probability 2/3 and c with probability 1/3. The notation $A\ [p]\ B$ denotes the mixture giving A with probability p and B otherwise.

Recall Independence:

$$X \succ Y \quad \text{if and only if} \quad X\ [p]\ Z \succ Y\ [p]\ Z \qquad (0 < p < 1).$$

(a) Express each O_i in the form $X\ [0.3]\ Y$, where X is either the certain outcome b or the prospect D, and Y is either the certain outcome b or the certain outcome c.

(b) Use these expressions and Independence to prove that the two stated strict preferences cannot both satisfy that axiom.

Solution.

(a)

$$O_1 = b\ [0.3]\ b, \quad O_2 = D\ [0.3]\ b, \quad O_3 = b\ [0.3]\ c, \quad O_4 = D\ [0.3]\ c.$$

For example, the mixture with D assigns $0.3(2/3) = 0.2$ to a and $0.3(1/3) = 0.1$ to c, in addition to its other branch.

(b) Applying strict Independence to $O_1 \succ O_2$ gives $b \succ D$. Applying it to $O_4 \succ O_3$ gives $D \succ b$. Both cannot hold, since strict preference is asymmetric. Therefore the two stated preferences cannot both satisfy Independence.