Optional Exercises (Week 5) — Solutions

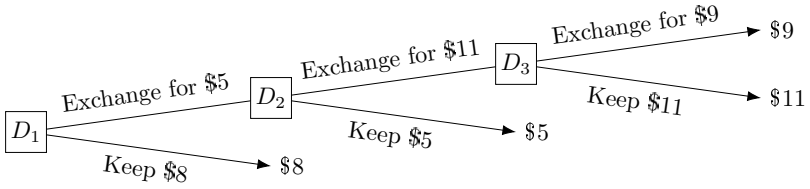
- **Myopic choice:** compare the holdings or prospects received immediately, ignoring later choices.
- **Naive choice:** select a continuation using current preferences, without anticipating later departures from it. At each later decision node, choose afresh.
- **Sophisticated choice:** work backwards, predicting later choices and comparing their consequences using the preferences held at the current decision node.
- **Resolute choice:** select a complete plan using the initial preferences and carry it out even if it later goes against what the agent's preferences would have been without the plan.

A complete plan specifies a choice at every decision node, including nodes it does not reach. The whole tree is known in advance. Squares are decision nodes; circles are chance nodes.

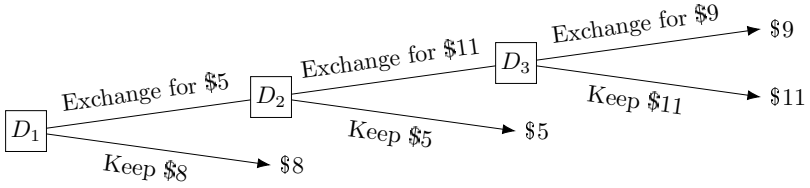
Question 1

An agent starts with \$8 and always prefers more money to less. Each exchange replaces the current holding by the stated amount. Keeping the current holding ends the sequence.

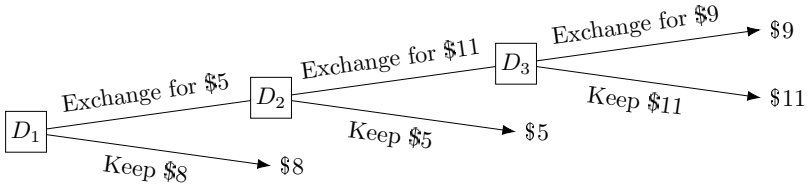
Myopic choice



Naive choice



Sophisticated choice



(a) On the first tree, shade the chosen branches or state the choice at every decision node, including any that will not be reached. State the final holding.

(b) Do the same for naive choice on the second tree.

(c) Do the same for sophisticated choice on the third tree.

(d) At D_1, which two holdings does the myopic agent compare? Which two final holdings does the naive agent compare? Which two final holdings does the sophisticated agent compare?

Solution. **(a)–(c)** The choices include those at unreached nodes.

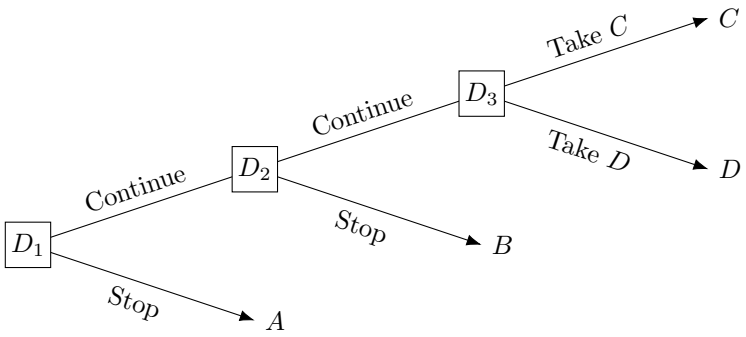
Procedure	At D_1	At D_2	At D_3	Final holding
Myopic	Keep \$8	Exchange for \$11	Keep \$11	\$8
Naive	Exchange for \$5	Exchange for \$11	Keep \$11	\$11
Sophisticated	Exchange for \$5	Exchange for \$11	Keep \$11	\$11

(d) Myopic choice compares \$8 with \$5. Naive choice compares \$8 with the best reachable final holding, \$11. Sophisticated choice also compares \$8 with \$11: it predicts keeping \$11 at D_3 and exchanging \$5 for \$11 at D_2.

Question 2

The agent's utilities for the four possible final outcomes are:

	A	B	C	D
At D_1	6	3	12	8
At D_2	0	10	1	7
At D_3	0	10	1	7



(a) Complete the table. Give the choices at D_2 and D_3 even when an earlier choice means they will not be reached.

Procedure	At D_1	At D_2	At D_3	Final outcome
Naive				
Sophisticated				
Resolute				

(b) For naive choice, write down the outcome anticipated at D_1 and the outcome actually obtained.

(c) At D_2, which final outcome will the resolute agent obtain by carrying out the initial plan? If instead they started at D_2 without a pre-existing plan, which final outcome would they plan to reach?

Solution. **(a)**

Procedure	At D_1	At D_2	At D_3	Final outcome
Naive	Continue	Stop	Take D	B
Sophisticated	Stop	Stop	Take D	A
Resolute	Continue	Continue	Take C	C

For sophisticated choice, D_3 gives D since $7 > 1$. At D_2, stopping gives B, preferred to that continuation since $10 > 7$. At D_1, stopping gives A with initial utility 6, rather than the predicted B with initial utility 3.

(b) Naive choice anticipates C at D_1, where its utility is 12. It actually obtains B, because at D_2 the agent prefers B to either C or D.

(c) Carrying out the initial plan gives C. Starting afresh at D_2, the agent would plan to obtain B: its utility 10 exceeds the utilities 1 and 7 of C and D.

Question 3

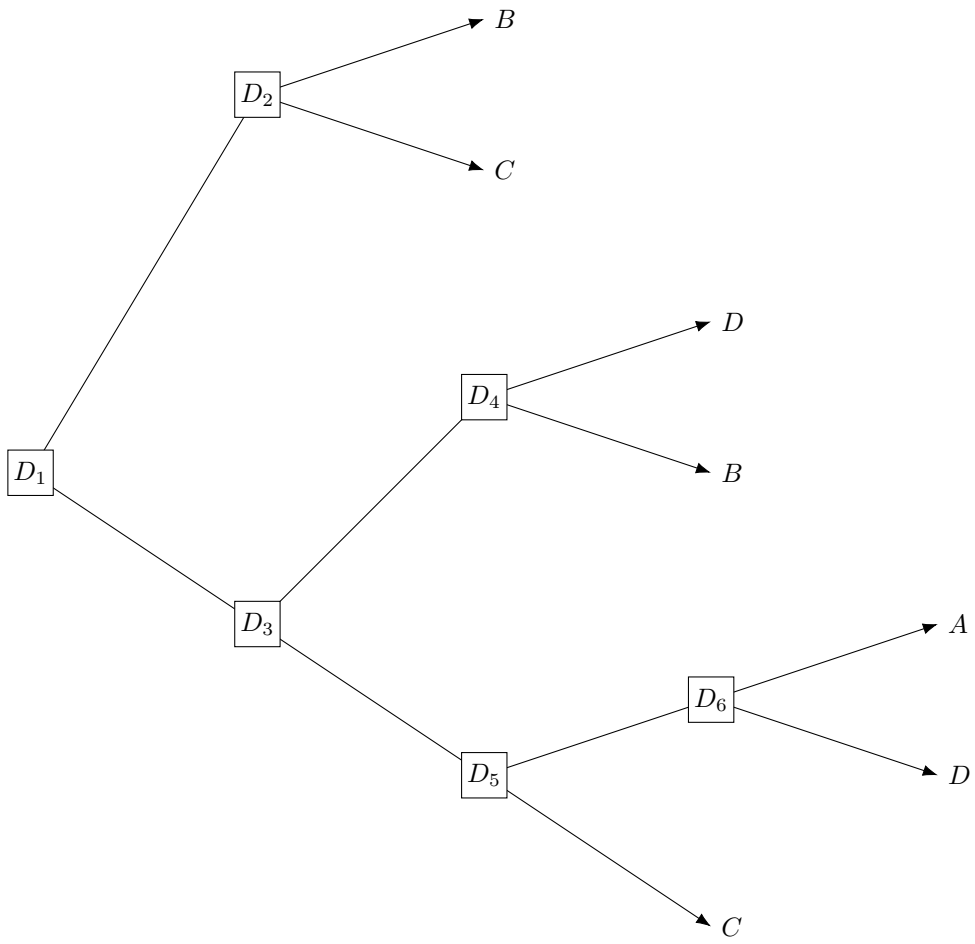
The agent's strict preferences remain the same throughout:

$$A \succ B, \qquad B \succ C, \qquad C \succ D, \qquad D \succ A,$$

and, for the remaining pairs, $C \succ A$ and $D \succ B$. From a pair of outcomes, the agent chooses the strictly preferred one. When more than two distinct outcomes are anticipated, the agent selects A if it is among them, and otherwise selects B. These rules apply in both parts. A sophisticated agent does not anticipate the results of a later choice that they predict they will not make.

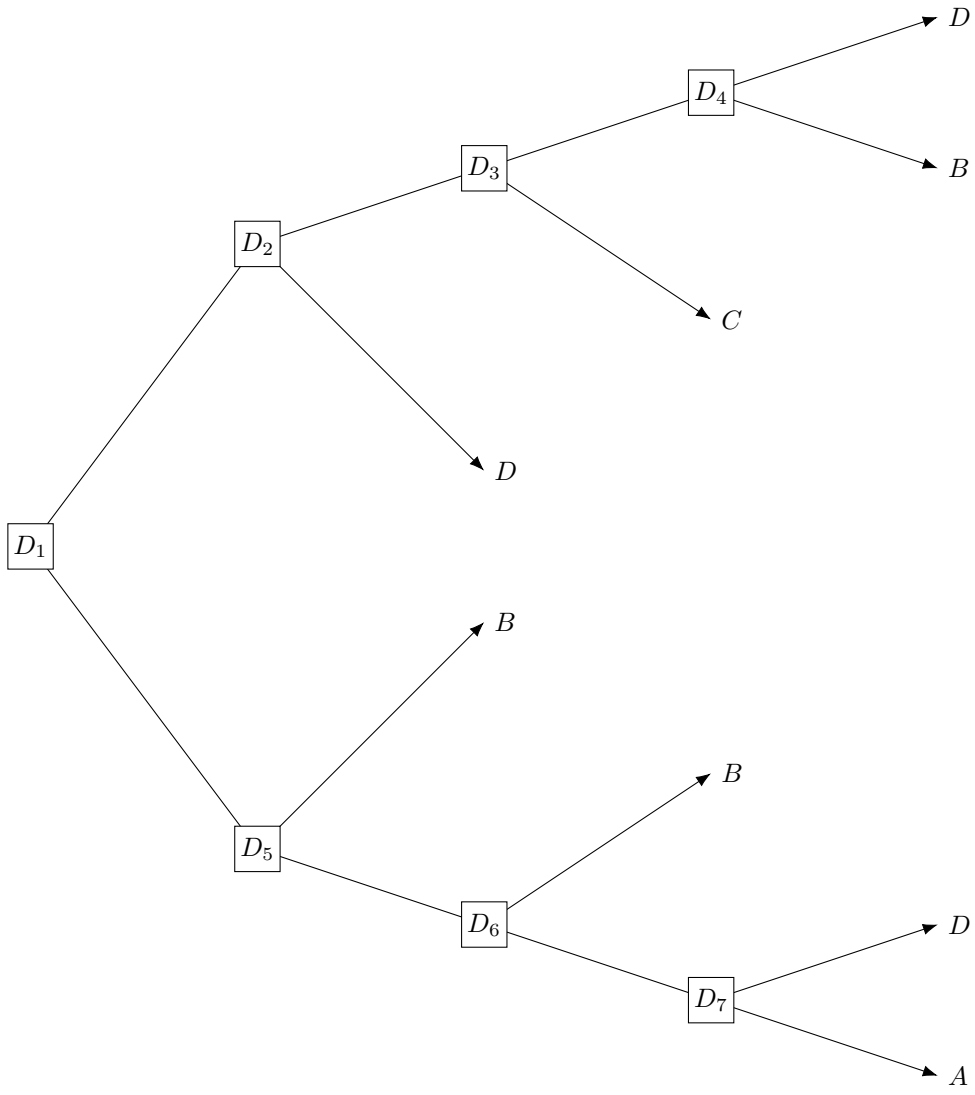
In each tree, label one branch at every decision node with each of N, S and R, for naive, sophisticated and resolute choice. A branch may carry several letters. For R, label the branches along the unique route to the outcome selected at D_1. Complete the R labels at unreached decision nodes in any way; these labels do not affect the route or final outcome. State the final outcome for each procedure.

(a)



Question 3 (continued)

(b) Use the same preferences and selection rules as in part (a).



Solution. UP and DOWN refer to the upper and lower branches. The tables include choices at unreached nodes.

(a)

Node	N	S	R
D_1	DOWN	UP	DOWN
D_2	UP	UP	UP
D_3	DOWN	DOWN	DOWN
D_4	UP	UP	UP
D_5	UP	DOWN	UP
D_6	DOWN	DOWN	UP

- Naive: $D_1 \to D_3 \to D_5 \to D_6$, ending at D.
- Sophisticated: $D_1 \to D_2$, ending at B.
- Resolute: $D_1 \to D_3 \to D_5 \to D_6$, ending at A.

Naive choice heads towards A until D_6, where it chooses $D \succ A$. For sophisticated choice, D_6 gives D, so D_5 selects $C \succ D$. Since D_4 gives D, D_3 also selects the continuation giving C. Finally D_2 gives B, so D_1 selects $B \succ C$. The resolute plan leads to A; its choices at D_2, D_4 do not affect that route.

(b)

Node	N	S	R
D_1	DOWN	UP	DOWN
D_2	UP	UP	UP
D_3	UP	DOWN	UP
D_4	UP	UP	DOWN
D_5	DOWN	DOWN	DOWN
D_6	DOWN	DOWN	DOWN
D_7	UP	UP	DOWN

- Naive: $D_1 \to D_5 \to D_6 \to D_7$, ending at D.
- Sophisticated: $D_1 \to D_2 \to D_3$, ending at C.
- Resolute: $D_1 \to D_5 \to D_6 \to D_7$, ending at A.

Naive choice heads towards A until D_7, where it chooses $D \succ A$. At unreached D_2, D_3, the available final outcomes are B, C, D, so it plans towards B under the stated selection rule.

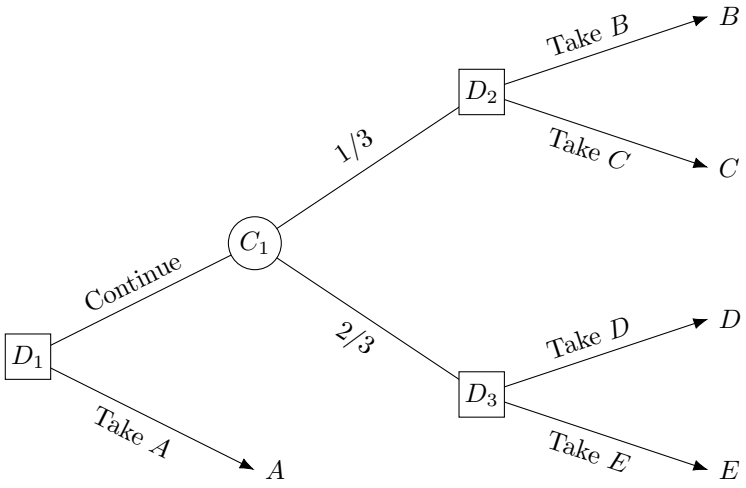
For sophisticated choice, D_4 gives D, D_3 selects $C \succ D$, and D_2 continues towards C. In the lower subtree, D_7 gives D, and D_6, D_5 both select $D \succ B$. Thus D_1 selects $C \succ D$. The initial resolute plan leads to A. If starting instead at D_2, the agent would select a plan leading to B: UP at D_2, UP at D_3, and DOWN at D_4. The table uses that plan for the unreached subtree.

Question 4

Write plans here as $(D_1: \ldots,\ D_2: \ldots,\ D_3: \ldots)$.

The probabilities on the chance branches are the agent's credences before the chance event is resolved. The agent evaluates prospects by expected utility. Utilities at the decision times are:

	A	B	C	D	E
At D_1	4	12	3	6	9
At D_2	0	1	8	10	2
At D_3	0	1	8	10	2



(a) Find a complete plan that maximises expected utility using the utilities as evaluated at D_1. Calculate its expected utility.

(b) For resolute choice, either mark the selected branches with **R**, or state the plan's choices in writing, including choices at decision nodes that will not be reached. Give the probability of each final outcome under the decision selected by the resolute agent at D_1.

(c) For sophisticated choice, either mark the chosen branches with **S**, or state the choices in writing, including choices at decision nodes that will not be reached. Calculate the initial expected utility of Continue under this prediction and give the resulting probability of each final outcome. A branch may carry both letters.

Solution.

(a) The plan is $(D_1:$ Continue, $D_2:$ Take B, $D_3:$ Take $E)$. Initially, B is preferred to C, and E to D. Its expected utility is

$$\tfrac{1}{3}(12) + \tfrac{2}{3}(9) = 10 > 4,$$

so it is preferred to taking A.

(b) Resolute choice follows that plan. In the order A, B, C, D, E, the final-outcome probabilities are $(0, \frac{1}{3}, 0, 0, \frac{2}{3})$.

(c) The sophisticated choices are $(D_1:$ Continue, $D_2:$ Take C, $D_3:$ Take $D)$. At D_2, $8 > 1$ favours C; at D_3, $10 > 2$ favours D. The initial expected utility of Continue is

$$\tfrac{1}{3}(3) + \tfrac{2}{3}(6) = 5 > 4.$$

The final-outcome probabilities, in the same order, are $(0, 0, \frac{1}{3}, \frac{2}{3}, 0)$.

Question 5

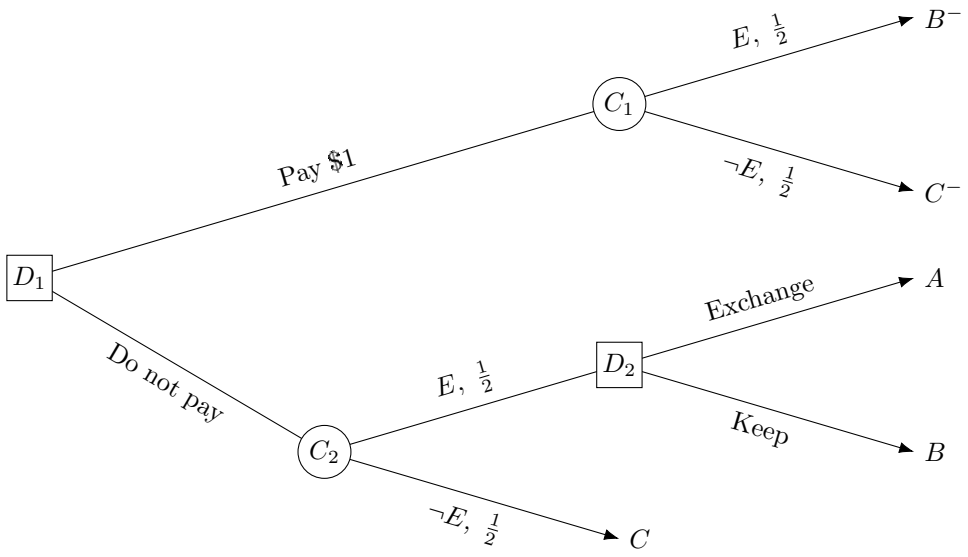
An agent initially holds a lottery $B[\frac{1}{2}]C$: they will receive B if an event E occurs and C otherwise. The agent's initial credence in E is $\frac{1}{2}$, as is its probability.

Before learning whether E occurs, the agent can pay \$1 to end all further offers. If they do not pay and E occurs, they will be offered a free exchange of B for A. If E does not occur, there is no further choice.

Write B^- and C^- for receiving B and C, respectively, with \$1 deducted. The agent's preferences are

$$A \succ B, \qquad B[\tfrac{1}{2}]C \succ A[\tfrac{1}{2}]C,$$
$$B[\tfrac{1}{2}]C \succ B^-[\tfrac{1}{2}]C^-, \qquad B^-[\tfrac{1}{2}]C^- \succ A[\tfrac{1}{2}]C.$$

These preferences do not change. Both chance nodes below concern the same event E.



(a) Given $A \succ B$, what preference between $A[\frac{1}{2}]C$ and $B[\frac{1}{2}]C$ would Independence require? Do the stated preferences satisfy this requirement?

(b) Suppose the agent follows sophisticated choice. What will they choose at D_2 if it is reached?

(c) Using your answer to (b), write down the lottery the sophisticated agent anticipates receiving if they do not pay at D_1. Which branch will they choose at D_1, and which lottery will they receive?

(d) Specify the choices at D_1 and D_2 that would preserve the original lottery $B[\frac{1}{2}]C$. Compare the outcome under this plan with the sophisticated agent's outcome if E occurs, and then if E does not occur. How much money does the sophisticated agent lose in each case relative to this plan?

Solution.

(a) Independence requires $A[\frac{1}{2}]C \succ B[\frac{1}{2}]C$. The given strict reverse preference violates this requirement.

(b) Exchange B for A, since $A \succ B$.

(c) Not paying leads to the anticipated lottery $A[\frac{1}{2}]C$. Since $B^-[\frac{1}{2}]C^- \succ A[\frac{1}{2}]C$, the agent pays \$1 and receives $B^-[\frac{1}{2}]C^-$.

(d) Choose Do not pay at D_1 and Keep at D_2. This preserves $B[\frac{1}{2}]C$. If E occurs, the preserving plan gives B, while sophisticated choice gives B^-. If $\neg E$ occurs, they give C and C^- respectively. The sophisticated agent loses \$1 in either case relative to that plan.

Question 6

State whether each statement is **true or false**. No proofs or explanations are required.

By an agent being *vulnerable to money pumps*, we mean that there exists a decision tree in which a cost-free plan would preserve the agent's initial holding, but following their preferences and dynamic decision rule leads to a guaranteed loss with no change in what they otherwise hold. The claims concern the money-pump arguments covered in the lecture.

(i) Any agent who follows naive choice is vulnerable to money pumps.

(ii) An agent can use resolute choice to avoid a money pump, even if her preferences are cyclic.

(iii) An agent who follows sophisticated choice is not vulnerable to money pumps, even if her preferences are cyclic.

(iv) If an agent's preferences satisfy acyclicity, they satisfy transitivity.

(v) An agent applies resolute choice by paying a small cost to prevent herself from having the ability to choose certain options at a later decision node.

(vi) Odysseus applied resolute choice in order to escape the sirens while still listening to their song.

Clarificatory note: in answering this question, you should imagine that Homer's epic poem, The Odyssey, is a factual account of real events.

(vii) Employing naive choice can block certain kinds of money pumps.

(viii) Money pump arguments are intended to establish that certain patterns of preferences are irrational.

(ix) Money pump arguments are intended to establish that following myopic choice is irrational.

(x) An agent can avoid all of the money pumps we have seen if they accept all of the axioms of Expected Utility Theory.

Solution.

(i) **False.** An agent who follows naive choice but also satisfies the axioms of expected utility theory is not vulnerable to money pumps.

(ii) **True.** Since a resolute chooser sticks to an optimal plan, and a plan to get money pumped is never optimal, they will always follow an alternative plan rather than get money pumped.

(iii) **False.** We saw in lectures and in the readings that there are money pumps for sophisticated choosers with cyclic preferences.

(iv) **False.** Acyclicity is a logically weaker requirement than transitivity: all transitive preferences are acyclic, but not all acyclic preferences are transitive. For example, the preference pattern $a \sim b$, $b \succ c$, $a \not\succ c$ is intransitive but acyclic.

(v) **False.** Resolute choosers follow a plan without needing to pay a cost to bind themselves.

(vi) **False.** No, Odysseus effectively "paid" to remove a later option. If he was successfully following resolute choice, he would have maintained his plan without binding himself.

(vii) **True.** Depending on how they are set up, some versions of the standard money pump for cyclic preferences can be blocked by naive choice. The upfront money pump for cyclic preferences, while effective against sophisticated choice, is also blocked by naive choice.

(viii) **True.** Yes, it is supposed to be irrational if following one's preferences causes oneself to poorly satisfy them in a sequential decision situation.

(ix) **False.** No, money pump arguments are intended to establish that preference patterns are irrational, not approaches to decision-making over time like myopic, naive, sophisticated or resolute choice.

(x) **True.** Yes, an agent who maximises expected utility and whose preferences do not change can reliably progress towards a prospect with maximum expected utility.